Equivariant Cohomology

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Contents

1 An introduction to equivariant cohmology by Loring Tu

- Video lectures: `https://www.youtube.com/watch?v=0hwDePh2RoY&list=PLQZfZKhc0kiAgfFYCdSQ3px6rHc9SMCXY&index=2&ab_channel=NCTSMathDivision`
- Book: *Introductory Lectures on Equivariant Cohomology* by Loring W. Tu.

1.1 Lecture 1: Overview

Equivariant cohomology is essentially the algebraic topology of a space with a group action.

Cohomology (with any kind of coefficients) is a functor $\{\text{topological spaces}, \text{continuous maps}\} \to$ **Rings**. It gives invariants of topological spaces.

In this course, not only topological spaces, but topological spaces with a group action.

Def. An action of group on a topological space X is a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ s.g. $1 \cdot x = x$, $g \cdot (h \cdot x) = (gh) \cdot x$, $\forall g, h \in G$.

Equivariant cohomology $H^*_G(\cdot)\colon \{G\text{-space}\} \to$ **Rings**.

De Rham theorem: For a C^∞ manifold M, there is an isomorphism $H^*(M; \mathbb{R}) \simeq H^*\{\Omega(M)\}$, where $\Omega(M)$ is the complex of C^∞ forms on M.

In equivariant cohomology, there is an analog of the de Rham theorem:

The equivariant de Rham theorem: let G be a Lie group and M a C^∞ G-manifold.

It is possible to construct a differential complex $\Omega_G(M)$ out of C^∞ forms on M and the Lie algebra $g \in G$, s.t. $H^*_G(M) = H^*\{\Omega_G(M)\}$. The complex $\Omega_G(M)$ is called the *Cartan complex.* the elements of the Cartan complex $\Omega_G(M)$ is the equivariant differential forms.

For example, if $G = S^1$, then $\Omega_{S^1}(M) = \{\sum \alpha_i u^i | \alpha_i \in \Omega(M)^{S^1}\}$.

(If X is in the Lie algebra of S^1, which is $\mathbb{R}$, or $T_1(S^1)$ (the tangent space at origin of S^1), u is the dual basis, i.e. the basis for $T_1^*(S^1)$)

Equivariant cohomology is very useful because it can be used to calcualte ordinary integrals on a manifold (which is usually very hard). We have

Equivariant localization theorem (Atiyah, Bott, Berligne, Vergne): Let G be a torus $(S^1 \times \cdots \times S^1)$ and M a compact oriented G-manifold with isolated G fixed points. If ω is an equivariantly closed form, then

$$\int_M \omega = \sum_{p \in M^G} \frac{r_p^* \omega}{e_G(\nu_p)},$$

where ν_p is the normal bundle of p (which is the tangent space), and e_G is the equivariant Euler class.

This theory gives a method for calculating the integral of an ordinary differential form.

The main theorems of this course is The equivariant de Rham theorem: and the Equivariant localization theorem.

1.2 Lecture 2: Definition of equivariant cohomology

Definition (G-space). A G-space X is a topological space X with continuous action of a topological group G.

Definition (G-equivariance). If X and Y are G-spaces, a *morphism* is a continuous map $f\colon X \to Y$ s.t.

$$f(g \cdot x) = g \cdot f(x), \quad \forall g \in G, x \in X.$$

Such a map is called G*-equivariant.*

Candidates for $H^*_G(-)$: 1) $H^*(X/G)$, where $X/G = \{G\text{-orbits}\}$.

Example: $G = \mathbb{Z}$ acts on $M = \mathbb{R}$ by $n \cdot x = x + n$, then $M/G = S^1$, $H^*(M/G) = H^*(S^1) = \begin{cases} \mathbb{R}, & \text{degree 1,2} \\ 0 & \text{otherwise} \end{cases}$

Example: $G = S^1$ acts on $M = S^2$ by rotation. $M/G = I$, so the quotient space cohomology is trivial, which is off of our expectation of the definition of equivariant cohomology.

Crucial difference between the two examples: in the 1st example the G action is *free*, where in the 2nd example it is not. "When you have a free action, and you take the quotient, you get something nice; otherwise, what you get can be something weird."

Definition (Free action). If G acts on X, the stabilizer $\mathrm{Stab}(x) := \{g \in G | g \cdot x = x\}$. The action is *free* if $\mathrm{Stab}(x) = \{1\} \forall x \in X$.

Every left action of G on X can be converted to a right action. (Suppose G acts on the left on X, then $x \cdot g := g^{-1} \cdot x$ suffices.)

If G acts on P and M on the left. Then the *diagonal action* of G on $P \times M$ is

$$g \cdot (p, m) := (g \cdot p, g \cdot m).$$

If G acts on P on the right but on M on the left, then the *diagonal action* is

$$g \cdot (p, m) := (p \cdot g^{-1}, g \cdot m).$$

Lemma. If G acts freely on P on the right, then no matter how it acts on M on the left, the diagonal action of G on $P \times M$ is free.

Proof: look at the stabilizer: suppose $g \cdot (p,m) = (p,m) \Leftrightarrow (p \cdot g^{-1}, g \cdot m) = (p,m) \Leftrightarrow p \cdot g^{-1} = p$ and $g \cdot m = m$, but as G acts freely on P so we must have $g = e$ from the first.

For any topological group G, there exists a contractible space P on which G acts freely. We denote such a space as EG (there can be many such spaces).

If P is a contractible space on which G acts freely, then $P \times M$ will have the same homotopy type as M, and G will act freely on $P \times M$. It turns out that such a space P exists for any topological group G, and it is deoted by EG.

Definition (Homotopy quotient, equivariant cohomology). . The *homotopy quotient* of M by G is defined as $M_G := (EG \times M)/G$, and we define *equivariant cohomology* as

$$H^*_G(M) := H^*(M_G).$$

(Need to prove that this definition is independent of the choice of EG.)

Let $G = S^1$. S acts on $\mathbb{C}^{n+1}$ by $\lambda(z_0, z_1, ..., z_n) = (\lambda z_0, ..., \lambda z_n)$. If $\sum |z_i|^2 = 1$, then it defines S^{2n+1}. S^1 acts on S^{2n+1}. Def: $S^{2n+1}/S^1 = \mathbb{C}P^n$. We have $S^1 \subset S^3 \subset S^5 \subset \cdots$, and taking the quotient of each place with respect to S^1 gives $\mathbb{C}P^1$, $\mathbb{C}P^2$,...

Let $S^\infty = \bigcup_{n=0}^\infty S^{2n+1}$. There is an action of S^1 on S^∞, which is a free action: $\lambda \cdot (z_0, ..., z_n) = (z_0, ..., z_n)$, i.e. $\lambda z_i = z_i$, since there's at least one j s.t. $z_j \neq 0$, so from $\lambda z_j = z_j$ we get $\lambda = 1$. Therefore, S^1 acts freely on S^∞.

Definition (Weakly contractible). A space X with $\pi_q(X, x_0) = 0$ for all $q \geq 0$ is called *weakly contractible.*

it's easy to show that this definition does not depend on the choice of x_0.

Theorem (Whitehead's theorem). If a continuous map $f \colon X \to Y$ of CW complexes induces an isomorphism in all homotopy groups π_q, then f is a homotopy equivalence.

Corollary. A weakly contractible CW complex X is contractible.

We will show that S^∞ is contractible by showing that S^∞ has vanishing homotopy groups at all positive degrees. See next lecture.

$\pi_1(X, x_0) = \{[\text{continuous maps } f(S^1, 1) \to (X, x_0)]\}$

$\pi_q(X, x_0) = \{[\text{continuous maps } f(S^q, (1, 0, ..., 0)) \to (X, x_0)]\}$

1.3 Lecture 3: Homotopy groups and CW complexes

Definition (Fiber bundle). A *fiber bundle* with fiber F is a subjection $\pi \colon E \to B$ which is locally a product $U \times F$, o.e. every point $b \in B$ has a neighborhood U s.t. there is a fiber-preserving homeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$.

Example: (1) Covering space $\pi \colon E \to B$; (ii) $\pi \colon \mathbb{R} \to S^1$ is a bundle with fiber $\mathbb{Z}$; (iii) $\pi \colon S^{2n+1} \to \mathbb{C}P^n$ is a fiber bundle with fiber S^1.

Theorem (Homotopy exact sequence of a bundle). Suppose $\rho \colon (E, x_0) \to (B, b_0)$ is a fiber bundle with fiber $F = \rho^{-1}(b_0)$. Assume B is path-connected. Let x_0 be a base point of F, and $i \colon (F, x_0) \to (E, x_0)$ the inclusion. Then $\exists$ exact sequence

$$\cdots \to \pi_k(F, x_0) \xrightarrow{i_*} \pi_k(E, x_0) \xrightarrow{\rho_*} \pi_k(B, x_0) \to \pi_{k-1}(F, x_0) \to \cdots \to F_0(F, x_0) \to \pi_0(E, x_0).$$

Example: $\pi_k(S^1)$. By the homotopy exact seuqnce of $\pi \colon \mathbb{R} \to S^1$ above, it is easy to get $\pi_k(S^1) = \begin{cases} 0 & k \geq 2, \\ \mathbb{Z}, & k = 1, \\ \{0\}, & k = 0. \end{cases}$

Def. (Attaching cells): let D^n be the n-dimensional closed unit disk. Let A be a topological space. $\phi \colon \partial D^n \to A$ the attaching map. X is obtained from A by attaching an n-cell via ϕ if $X = (A \amalg D)/\sim$, where $x \in \partial D^n \sim \phi(x) \in A$.

Denote $e^n = \mathrm{int}(D^n)$ to be the (image of the) interior of D^n in $X = (A \amalg D^n)/\sim$. We write $X = A \cup_\phi e^n = A \cup e^n$. We can attach infinitely many cells all at once:

$$X = (A \amalg (\amalg_\lambda D^n_\lambda)) / \sim = A \cup \left(\bigcup_\lambda e^n_\lambda \right),$$

Definition (CW complex). A *CW complex* is a Hausdorff space X with an increasing sequence of closed subspaces $X^0 \subset X^1 \subset X^2 \subset \cdots$ s.t. (i) X^0 is a discrete set of points, (ii) for $n \geq 1$, X^n is obtained from X^{n-1} by attaching n-cells e^n_λ, (iii) X has the *weak* topology: S is closed in X if and only if $S \cap X^n$ is closed in X^n, for all $n \geq 0$.

Theorem (closure-finite condition). The closure of each cell in a CW complex conains only finitely many cells of lower dimensions.

Example. $S^\infty := \bigcup_{n=0}^{\infty} S^n = \bigcup_{k=0}^{\infty} S^{2k+1}$ is a CW complex with weak topology. Now we show S^∞ is contractible: last time we proved $\pi_k(S^n) = 0$ for $k < n$.

Theorem: $\pi_k(S^\infty) = 0$ $\forall k$.

(Proof: S^∞ has the same homotopy type as the "telescope", see te video lecture. It's shown in the lecture that the telescope defines a deformation retraction of the telescope to S^∞. Next, it's shown in the lecture that The telescope has a projection to $\mathbb{R}$: $\pi\colon$ Telescope $\to \mathbb{R}$. Since S^k is compact, $(\pi \circ f)(S^k)$ is compact in $\mathbb{R}$, so is closed and bounded. So it lies in $[0, N]$ for some $N \in \mathbb{Z}^+$. Thus, $f(S^k) \subset \pi^{-1}([0, N])$ = a finite telescope, ending in S^N, which has the same homotopy type as S^N. One can choose $N > k$, Then $f(S^k)$ is null-homotopic. This proves that $\pi_k(S^\infty) = 0$ $\forall k$.)

This shows the CW complex S^∞ is weakly contractible so is also contractible (by the corollary in the last lecture).

1.4 Lecture 4: Principal bundles

[Large part of the lecture is missing from the video.]

Definition (Principal G-bundle). A *principal G-bundle* is a fiber bundle $\pi\colon P \to B$ with fiber G and an open cover $\{(U, \phi_U)\}$ of B s.t. (i) G acts freely on the right on P; (ii) for each U, the fiber-preserving homeomorphism $\phi_U : \pi^{-1}(U) \to U \times G$, where G acts on the right on $U \times G$ by $(u, x)g = (u, xg)$, is G-equivariant.

Note that the base space B has trivial G action.

Definition (G-bundle map). Let $P \to M$ and $E \to B$ be two principal G-bundles. A *morphism* of principal G-bundles, or a G-bundle map, from $P \to M$ to $E \to B$ is a morphism of fiber bundles

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "h"] & B \end{tikzcd}$$

in which $f\colon P \to E$ is G-equivariant.

Definition (Pullback bundle). Let $\pi\colon E \to B$ be a fiber bundle with fiber F and $h\colon M \to B$ a continuous map. the total space h^*E of the *pullback bundle* is defined as $h^*E := (\{(m, e) \in M \times E | h(m) = \pi(e)\}$. Define the projections p_1 and p_2 of $M \times E$ to M and E, respectively, then the pullback bundle is a bundle that fits into the diagram

$$\begin{tikzcd} h^*E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "h"] & B \end{tikzcd}.$$

Proposition. The first projection map $p_1 : h^*E \to M$, $p_1(m, e) = m$, is a fiber bundle with fiber F.

(Suffices to show that the pullback $h^*(U \times F)$ of a product bundle over U is a product bundle $h^{-1}(U) \times F$ over $h^{-1}(U)$. This is indeed true, as we have the isomorphism $h^*(U \times F) \xrightarrow{\simeq} h^{-1}(U) \times F$ given by $(m, h(m), f) \mapsto (m, f)$.)

Proposition (Universal property of the pullback). Given a bundle map $\begin{matrix} P & \xrightarrow{q_2} & E \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{h} & B \end{matrix}$, there is a unique bundle map $\phi\colon P \to h^*E$ over M such that the following diagram commtes:

$$\begin{tikzcd} \exists!\ P \arrow[dr, "\phi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & h^*E \arrow[r, "p_2"] \arrow[d, "p_1"] & E \arrow[d, "\pi"] \\ & M \arrow[r, "h"] & B \end{tikzcd}$$

Cartan's mixing space and diagram (§4.3):

Definition (Cartan mixing space). If $\alpha\colon P \to B$ is a principle G-bundle, and M a left G-space. Then one can form the *mixing space* (Borel construction) $P \times_G M := (P \times M)/G = (P \times M)/\sim$, where $(p, m) \sim (p, m) \cdot g = (pg, g^{-1}m)$. If $y = g^{-1}m$, then $m = gy$, so $(p, gy) \sim (pg, y)$.

Let $[p,m] :=$ equivalence class of (p,m).
Define $\tau_1\colon P\times_G M \to B$ by $\tau_1([p,m]) = \alpha(p)$. ($\tau_1$ is well defined: $\tau_1([p,m]) = \tau_1([pg, g^{-1}m]) = \alpha(pg) = \alpha(p)$.)

Proposition (4.5). $\tau_1\colon P\times_G M \to B$ is a fiber bundle with fiber M.

(The claim that the fiber is M can be rationalized by setting P to be a product bundle $P = B\times G$, in which case $P\times_G M \simeq (B\times G)\times_G M \simeq B\times M$ so this is indeed a fiber bundle, with base space B and fiber M. For why we have $(B\times G)\times_G M = B\times M$, see the proof below.)

[Proof: due to the local trivialization property of fiber bundle, we only have to prove the case for when P is a direct product bundle: $P = U\times G$. For $U = B$. (Otherwise we choose an open set $U\subset B$ on which $\pi^{-1}(U) = U\times G$ and proceed in the same way.) We have $\tau_1^{-1}(U) = \alpha^{-1}(U)\times_G M \simeq (U\times G)\times_G M = U\times M$, completing the proof. Note that here $\tau_1^{-1}(U) = \alpha^{-1}(U)\times_G M$ follows from the definition of $P\times_G M$; $\alpha^{-1}(U)\times_G M \simeq (U\times G)\times_G M$ follows the functoriality of $(-)\times_G M$, and $(U\times G)\times_G M \simeq U\times M$ can be proven using the explicit map $[(u,g),m]\mapsto (u,gm)$ and show that it has an inverse $(u,gm)\mapsto [(u,id),gm]$.]

This proposition can be nicely characterized by the *Cartan's mixing diagram:*

$$\begin{tikzcd} P \arrow[d, "\alpha"] & P\times M \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\beta"] & M \arrow[d, "\gamma"] \\ B & P\times_G M \arrow[l, "\tau_1"'] \arrow[r, "\tau_2"] & M/G, \end{tikzcd} \tag{1.1}$$

where (by the hypothesis of the proposition) $P\to B$ is a principal bundle. On general setting, one can prove that if $P\to B$ is a principal bundle then $\tau_1\colon P\times_G M\to B$ is a fiber bundle. Furthermore, the fact that the fiber is M (what's in the upper-right corner) comes from the fact that G acts freely on P ((i) in the definition of principal G-bundle).

Summary: in Cartan's mixing diagram, if the vertical map $P\xrightarrow{\alpha} B$ is a principal bundle, then the lower horizontal map $P\times_G M\xrightarrow{\tau_1} B$ is a fiber bundle, whose fiber is M in the far corner of the other square. This pattern will be repeatedly used later.

Theorem (4.10). In the category of CW complexes, suppose G acts on the left on M, and E, E' are weakly contractible spaces on which G acts freely. Then $E\times_G M$ and $E'\times_G M$ are weakly homotopy equivalent.

(This shows that equivalent cohomology is well-defined, independent of the contractible space on which G act freely that you choose.)

1.5 Lecture 5: Universal bundles

In defining $H_G^*(M)$, we chose a contractible space E on which G acts freely.

Such a space is the *total space* of a *universal bundle.*

$$\begin{tikzcd} P \arrow[r, "\simeq"] \arrow[dr] & h^*E \arrow[r] \arrow[d] & E \arrow[d] \\ & X \arrow[r, "h"] & BG \end{tikzcd}$$

Definition (Universal G-bundle). A *universal G-bundle* is a principal G-bundle $EG\to BG$ if (i) for any principle G-bundle P over a CW complex X, $\exists$ a map $h\colon X\to BG$ s.t. $P\simeq h^*(EG)$; (ii) if $h_0, h_1\colon X\to B$ are two maps s.t. $h_0^*(EG)\cong h_1^*(EG)$, then h_0 and h_1 are homotopic.

Theorem (Homotopic maps pull back to isomorphic bundles). If $h_0, h_1\colon X\to B$ are homotopic, and $E\to B$ is a principal G-bundle, then $h_0^*E\simeq h_1^*E$.

Let $P_B(X) = \{$isomorphism classes of principle G-bundles over $X\}$. The above theorem says that: Fixing a universal bundle $EG\to BG$, the map

$$\varphi\colon [X, BG]\to P_G(X)$$

is well defined. Here $[X,BG] = \{$homotopy classes of maps $h\colon X\to BG\}$.

(i) in the definition of universal G-bundle $\Leftrightarrow$ surjectivity of φ.

(ii) in the definition of universal G-bundle $\Leftrightarrow$ injectivity of φ.

So $\varphi\colon [X,BG]\to P_G(X)$ is a (set-theoretic) bijection.

$(h\colon X\to BG)\mapsto h^*(EG)$.

$P_G(-)$ is a contravariant function on CW complexes; $[,BG]$ is also a contravariant functor, and $P_G(-) = [-,BG]$ as functors, i.e. $P_G(-)$ is a *representable* functor.

Definition (Classifying slace). BG is called a *classifying space* for G.

Example. $S^\infty \to \mathbb{C}P^\infty$, is a universal S^1-bundle (S^∞ is contractible), i.e. $ES^1 = \mathbb{C}P^\infty$, and $BS^1 = \mathbb{C}P^\infty$.

Theorem. A principal G-bundle $E \to B$ is universal in the category of CW complexes if E is weakly contractible.

Grassmannian is the set of all planes in Eucliean space;

(Recall that every compact Lie group is a subgroup of the orthogonal group. All frames – called Stiefel variety. Replace S^∞ with the infinite Stiefel variety will give the unversal bundle for any compact Lie group.)

Example. $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ is a universal $\mathbb{Z}$-bundle, i.e. $E\mathbb{Z} = \mathbb{R}$, and $B\mathbb{Z} = S^1$.

Below we want to show that equivariant cohomology is well defined.

Using the Cartan mixing diagram in Eq. (1.1), with $P \equiv E$: as $E \to B$ is a principal G-bundle, and M is a left G-space, there we proved $\tau_1\colon (E \times M)/G \to B$ is a fiber bundle with fiber M.

$E \times M \to (E \times M)/G$ is a principal G-bundle.

Lemma. If E is a weakly contractible space on which G acts freely, and $P \to B$ is a principal G-bundle, then $E \times P/G \sim P/G = B$ (have the same homotopy type).

(Proof: we have another Cartan's mixing diagram

$$\begin{array}{ccccc} E & \xleftarrow{\pi_1} & E \times P & \xrightarrow{\pi_2} & P \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ B & \xleftarrow{\tau_1} & E \times P/G & \xrightarrow{\tau_2} & P/G \end{array} \tag{1.2}$$

By homotopy exact sequence of a fiber bundle,

$\pi_k(E) \to \pi_k(E \times P/G) \to \pi_k(P/G) \to \cdots$,

we prove the lemma.)

1.6 Lecture 6: Equivariant cohomology, spectral sequences

First, continue the proof in the last lecture:

Fact [Hatcher, Pro 4.21] Weak homotopy equivalence $f\colon X \to Y$ (i.e. $f_*\colon \pi_k(X, x_0) \xrightarrow{\simeq} \pi_k(Y, f(x_0))$, $\forall k$) induces an isomorphism in homology $f_*\colon H_k(X, A) \xrightarrow{\simeq} H_k(Y, A)$ and cohomology $f^*\colon H^k(Y, A) \xrightarrow{\simeq} H^k(X, A)$ $\forall$ k and all coefficient groups A.

Suppose E_1 and E_2 are contractible spaces on which G acts freely. So that $E_1 \to E_1/G$, and $E_2 \to E_2/G$ are principal G-bundles.

Let $P = E_2 \times M$. We know $E_2 \times M \to (E_2 \times M)/G$ is a principal G bundle Applying lemma to E_1 and P, we get $E_1 \times (E_2 \times M)/G \sim E_2 \times M/G$ (weakly homotopy equivalent). By symmetry, $E_2 \times (E_1 \times M)/G \sim E_2 \times M/G$. But $E_1 \times (E_2 \times M)/G$ and $E_2 \times (E_1 \times m)/G$ are homeomorphic (just exchanging coordiates), they are weakly homotopy equivalent. Therefore $E_1 \times M/G \sim E_2 \times M/G$. By [Hatcher, Prop 4.21), we have $H^*(E_1 \times M/G) \simeq H^*(E_2 \times M/G)$ for any coefficient.

This proves that equivariant cohomology is well defined, independent of the choice of E.

(For CW complexes, if two spaces are weakly homotopy equivalent, then they are homotopy equivalent (Whitehead theorem).)

Now: spectral sequences

Spectral sequences: referred to as "*less digestible aspect of algebraic topology*" by Raoul Bott.

A differential group is a pair $(\mathcal{E}, d)$ where $\mathcal{E}$ is an abelian group and $d\colon \mathcal{E} \to \mathcal{E}$ is a group homomorphism s.t. $d^2 = 0$, so $\mathrm{im} d \subset \ker d$.

A *spectral sequence* is a sequence $\{(E_r, d_r)\}$ of differential groups s.t. $E_r = H^*(E_{r-1}, d_{r-1})$ for $r \geq 1$.

We assume $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$, and usually we assue $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$. $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. This means, for fixed (p, q), if $r \geq q + 2$, then d_r is a zero map, then $E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots = E_\infty^{p,q}$, where we called the stationary value $E_\infty^{p,q}$.

(We have $E_{r+1} = H^*(E_r, d_r) = \frac{\ker d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}}{\mathrm{im} d_r\colon E_r^{p-r,q+r-1} \to E_r^{p,q}}$)

A *filtration* on an abelian group M is a decreasing sequence of subgroups $M = D_0 \supset D_1 \supset D_2 \supset \cdots$, the *associated graded group* of $\{D_i\}$ is $GM = \frac{D_0}{D_1} \oplus \frac{D_1}{D_2} \oplus + \cdots$

If $E = \bigoplus_{p,q\in\mathbb{Z},p,q\geq 0} E^{p,q}$ then the filtration by p is $D_p = \bigoplus_{i \geq p, q\geq 0} E^{i,q}$.

Leray's theorem: see next lecture.

Let $\pi\colon E \to B$ be a fiber bundle with fiber F over a simply connected basis space B (the original spectral sequence is more general and did not assume simply connectedness; here we assume simply connected for simplicity). Assume that in every dimension n, $H^n(F)$ is of finite rank and free. Then $\exists$ a spectral sequence $\{E_r, d_r)\}$, with

$$E^{p,q} = H^p(B) \otimes H^q(F),$$

and a filtration $\{D_i\}$ on $H^*(E)$ s.t. $E_\infty = \bigoplus_{p,q} E_\infty^{p,q} \simeq GH^*(E)$, i.e.

1.7 Lecture 7: Computation using spectral sequence

Theorem (Leray's theorem). Let $\pi \colon E \to B$ be a fiber bundle with fiber F over a simply connected base space B. Assume $H^n(F)$ is free, of finite rank, for any $n \geq 0$. Then there exists spectral sequence $\{(E_r, d_r)\}$ with

$$E_2^{p,q} = H^p(B) \otimes H^q(F),$$

which is an equality as rings, and a filtration $\{D_i\}$ on $H^*(E)$ s.t. $E_\infty = GH^*(E)$. Moreover, $d_r : E_r \to E_r$ is an antiderivation, i.e. $d_r(\alpha\beta) = (d_r\alpha)\beta + (-1)^{\deg \alpha}\alpha d\beta$.

"A filtration $\{D_i\}$ on $H^*(E)$ s.t. $E_\infty = GH^*(E)$" means that there is a filtration $H^*(E) = D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_n \supset \cdots$, $GH^*(E) = \frac{D_0}{D_1} \oplus \frac{D_1}{D_2} \oplus \frac{D_2}{D_3} \oplus \cdots$. For each n, there is an induced filtrartion $\{D_i \cap H^n\} \subset H^n(E) := H^n$ s.t. $H^n(E) = (D_0 \cap H^n) \supset (D_1 \cap H^n) \supset (D_2 \cap H^n \supset) \cdots$, where $E_\infty^{0,n} = D_0 \cap H^n / D_1 \cap H^n$, $E_\infty^{1,n-1} = D_1 \cap H^n / D_2 \cap H^n$, ...

(Example: $G \supset \mathbb{Z}_2 \supset 0$, with $G/\mathbb{Z}_2 = \mathbb{Z}_2$, then G can still be $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

(Lemma: If $0 \to A \to B \to C$ is an exact sequence of abelian groups, and C is free, then $B \simeq A \oplus C$.)

Example: $H^*(\mathbb{C}P^2)$. Use $\begin{array}{c} S^1 \longrightarrow S^5 \\ \downarrow \\ \mathbb{C}P^2 \end{array}$, the homotopy exact sequence $\to \pi_1(S^1) \to \underbrace{\pi_1(S^5)}_{=0} \xrightarrow{\to 0} \pi_1(\mathbb{C}P^2) \xrightarrow{\simeq} \pi_0(S^1) \to \pi_0(S^5)$ says $\pi_1(\mathbb{C}P^2) = 0$, so $\mathbb{C}P^2$ is simply connected:

Then we can apply Leray's theorem and we have $E_2 = H^*(\mathbb{C}P^2) \otimes H^*(S^1)$.

We have $H^0(S^1) = \langle 1 \rangle$, and $H^1(S^1) = \langle x \rangle$, so the 0th column is $E_2^{0,q} = H^0(\mathbb{C}P^2) \otimes H^q(S^1) = \mathbb{Z} \otimes H^q(S^1) = H^q(S^1)$, where we used $\mathbb{Z} \otimes A = A$. So we have

$$E_2 = \begin{array}{c|ccccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q=2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ q=1 & x & ? & ? & ? & ? & 0 & \cdots \\ q=0 & 1 & ? & ? & ? & ? & 0 & \cdots \\ \hline & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & \cdots \end{array}$$

Considering the differentials, we have $E_3 = E_4 = \cdots = E_\infty = GH^*(S^5) = \mathbb{Z}$ when the degree is 0 and 5, and vanishes otherwise.

On $H^5(S^5)$, there is a filtration $H^5(S^5) = (D_0 \cap H^5) \supset (D_1 \cap H^5) \subset (D_2 \cap H^5) \cdots \cdots D_5 \cap H^5) \supset 0$, where $E_\infty^{0,5} = \frac{D_0 \cap H^5}{D_1 \cap H^5}$, $E_\infty^{1,4} = \frac{D_1 \cap H^5}{D_4 \cap H^5}$, $E_\infty^{2,3} = \frac{D_2 \cap H^5}{D_3 \cap H^5}$, and so on.

and we have $H^0(S^4) = \langle 1 \rangle$ and $H^4(S^4) = \langle u \rangle$, where all other degrees of $H^*(S^4) = 0$. So we have

$$E_3 = \begin{array}{c|ccccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q=2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ q=1 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & \cdots \\ q=0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & \cdots \end{array}$$

Using d_2, we see that there must be a u in the $(p,q) = (2,0)$ entry. Using the tensor product structure, we know that $(p,q) = (2,1)$ entry has ux. Again using d_2 we see that there must be a u^2 in the $(p,q) = (4,0)$ entry, then the tensor product structure says there's a $u^2 x$ in the $(p,q) = (4,1)$ entry. So we have

$$E_2 = \begin{array}{c|ccccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q=2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ q=1 & x & 0 & ux & 0 & u^2x & 0 & \cdots \\ q=0 & 1 & 0 & u & 0 & u^2 & 0 & \cdots \\ \hline & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & \cdots \end{array}$$

So

$$H^*(\mathbb{C}P^2) = \mathbb{Z} \oplus \mathbb{Z}u \oplus \mathbb{Z}u^2 = \mathbb{Z}[u]/(u^3).$$

1.8 Lecture 8: Equivariant cohomology of S^2 under rotation

$G = S^1$, $M = S^2$, where G acts on M as a rotation along the polar axis. We want to compute

$$H^*_{S^1}(S^2) = H^*((S^2)_{S^1}) = H^*((ES^1 \times S^2)/S^1) = H^*(S^\infty \times_{S^1} S^2).$$

By Cartan's mixing diagram, we have

$$\begin{tikzcd} S^\infty & S^\infty \times S^2 \arrow[l] \arrow[r] \arrow[d] & S^2 \arrow[d] \\ \mathbb{C}P^\infty \arrow[u, leftarrow] & S^\infty \times_{S^1} S^2 \arrow[l] \arrow[r] & S^2/S^1 \end{tikzcd} \tag{1.3}$$

there is a fiber bundle

$$\begin{tikzcd} M \arrow[r] & M_G \arrow[d] \\ & BG \end{tikzcd} = \begin{tikzcd} S^2 \arrow[r] & S^\infty \times_{S^1} S^2 \arrow[d] \\ & \mathbb{C}\mathbb{P}^\infty \end{tikzcd}$$

Fact: $H^*(\mathbb{C}P^n) = \mathbb{Z}[u]/(u^{n+1})$, then $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[u]$.

By Leray, $E_2^{p,q} = H^p(\mathbb{C}P^\infty) \otimes H^q(S^2)$

$$E_2 = \begin{array}{c|cccccccc} q=3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ q=2 & y & 0 & uy & 0 & u^2y & 0 & u^3y & \cdots \\ q=1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ q=0 & 1 & 0 & u & 0 & u^2 & 0 & u^3 & \cdots \\ \hline & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & \cdots \end{array}$$

Let's easy to see that $d_2 = d_3 = 0$, as well as the differential on later pages. So $E_2 = E_3 = \cdots = E_\infty$.

This shows that $E_\infty = GH^*((S^2)_{S^1})$, with

$$H^0((S^2)_{S^1}) = \mathbb{Z}, \quad H^1((S^2)_{S^1}) = 0, \quad H^2((S^2)_{S^1}) = (D_0 \cap H^2) \supset (D_1 \cap H^2) \supset (D_2 \cap H^2) \supset 0,$$

where $E_\infty^{0,2} = \frac{D_0 \cap H^2}{D_1 \cap H^2} = \mathbb{Z}y$, $E_\infty^{1,1} = \frac{D_1 \cap H^2}{D_2 \cap H^2} = 0$, and $E_\infty^{2,0} = \frac{D_2 \cap H^2}{D_3 \cap H^2} = \mathbb{Z}u$, therefore we have an exact sequence

$$0 \to \mathbb{Z}u \to H^2 \to \mathbb{Z}y \to 0,$$

Since $\mathbb{Z}y$ is free, $H^2((S^2)_{S^1}) = \mathbb{Z}u \oplus \mathbb{Z}y$.

Then we have $H^3((S^2)_{S^1}) = 0$, and $H^4((S^2)_{S^1}) = \mathbb{Z}uy \oplus \mathbb{Z}u^2$.

In general, $H^{\text{odd}}((S^2)_{S^1}) = 0$, $H^{2n}((S^2)_{S^1}) = \mathbb{Z}u^{n-1}y \oplus \mathbb{Z}u^n$.

So

$$H^*_{S^1}(S^2) = \mathbb{Z}[u] \oplus \mathbb{Z}[u]y = \mathbb{Z}[u,y]/(y^2 = auy + bu^2), \tag{1.4}$$

for some a, b. where deg u = deg $v = 2$, as abelian groups.

[We will find the coefficients a, b in Lecture 29, after introducing the Borel localization theorem.]

We will compute the cohomology of the space $(S^2)_{S^1}$ directly in the next lecture. Before that, let's introduce some general results:

Theorem. If G acts on M with at least one fixed point, then $H^*(BG)$ injects into $H^*_G(M)$.

Proof (This was actually done in lecture 10): The inclusion map $i\colon \{p\} \hookrightarrow M$ is a G-map if p is a fixed point. Let $\pi\colon M \to \{p\}$ be the constant map. Then $\pi \circ i = \mathbb{I}\colon \{p\} \to \{p\}$. By functoriality (see lecture 10), $i_G^* \circ \pi_G^* = \mathbb{I}^*\colon H^*_G(\{p\}) \to H^*_G(M) \to H^*_G((\{p\})$, hence $\pi_G^*\colon H^*_G(\{p\}) = H^*(BG) \to H^*_G(M)$ is injective.

We need to develop some tools to figure out what the ring structure is.

General theorems about equivariant cohomology:

N, M be left G-spaces. If $f\colon N \to M$ is equivariant, then there is an induced map $f_G\colon N_G \to M_G$, given by

$EG \times N \to EG \times M$, gives $[e,n) \mapsto [e, f(n)]$,

$EG \times_G N \to EG \times_G M$, $[eg, g^{-1}n] \mapsto [eg, f^(g^{-1}n)] = [eg, g^{-1}f(n)]$,

Consider $f\colon M \to pt$, which is G-equivariant.

$pt_G = EG \times pt/G = EG/G = BG$. f_G induces a map in cohomology $\begin{tikzcd} H^*(pt_G) \arrow[r] \arrow[d, equal] & H^*(M_G) \arrow[d, equal] \\ H^*(BG) & H^*_G(M) \end{tikzcd}$, This makes $H_G^*(M)$ into an $H^*(BG)$-module, so $H_G^*(M)$ is an $H^*(BG)$-algebra.

BG is the base of a universal bundle $EG \to B$ for G, and is called *the classifying space* for G.

Examples: $BS^1 = \mathbb{C}P^\infty$, $B\mathbb{Z} = S^1$.

Example $BO(k)$?

Definition (Stiefel varieties). $V(k,n) = \{\text{orthonomal } k \text{ frames in } \mathbb{R}^n\}$; k-frame is an ordered set of k-linear independent vectors. A k-frame spans a k-plane.

So there exists a map $V(k,n) \xrightarrow{G} (h,n)$, whose fiber is all the orthogonal bases of a k-plane, i.e. fiber $= O(k)$.

$V(1,n) = \{\text{unit vectors in } \mathbb{R}^n\} = S^{n-1}$

$G(1,n) = \mathbb{R}P^{n-1}$

1.9 Lecture 9: General properties of equivariant cohomology

Proposition. If a topological group G acts freely on a topological space M s.t. $M \to M/G$ is a principal G-bundle, then M_G is weakly homotopy equivalent to M/G.

(Recall that action is free $\Leftrightarrow$ any point has trivial stabilizer group; weakly homotopy equivalent $\Leftrightarrow$ homotopy group agree at all degrees.)

Proof: By Cartan's mixing diagral,

$$\begin{tikzcd}
EG \arrow[d] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\
BG & M_G \arrow[l] \arrow[r] & M/G
\end{tikzcd}$$

Since $M \to M/G$ is a principal G-bundle, $M_G \to M/G$ is a fiber bundle with fiber EG. Then, by the homotopy exact sequence of the fiber bundle, $\cdots \to \underbrace{\pi_k(EG)}_{=0} \to \pi_k(M_G) \xrightarrow{\simeq} \pi_k(M/G) \to \pi_{k-1}(EG) \to \cdots$ where $\pi_k(EG) = 0$ (for $k > 0$) as EG is contractible.

Example. S^1 acts on S^1 by $\lambda \cdot x = \lambda x$, where $\lambda, x \in S^1 \in \mathbb{C}$. For each $x \in S^1$, $\lambda x = x \Rightarrow \lambda = 1$, i.e. the action is free. Then, the proposition above says that $(S^1)_{S^1}$ is weakly homotopy equivalent to $S^1/S^1 = pt$.

To further show that $(S^1)_{S^1}$ is homotopy equivalent to $S^1/S^1 = pt$ (using Whitehead's theorem), we need to show that $(S^1)_{S^1}$ is a CW complex. We have $(S^1)_{S^1} = (ES^1 \times S^1)/S^1 = (S^\infty \times S^1)/S^1 \to S^\infty$, by $[e,s] \mapsto es$ which has an inverse map $[e] \mapsto [e,1]$ (so $[e,s] \mapsto [es] \mapsto [es,1] = [e,s]$, so $(S^1)_{S^1} = S^\infty$, which is a CW complex, so by Whitehead's theorem $(S^1)_{S^1}$ has the homotopy type of a point.

Example. Now, going back to the example in the last lecture, $(S^2)_{S^1}$. By the Cartan's mixing diagram in (1.3), the homotopy quotient $(S^2)_{S^1} \to BS^1 = \mathbb{C}P^\infty$ is a fiber bundle with fiber S^2.

Another picture: The orbit space is $S^2/S^1 = [-1,1]$; if we take out the north and south poles p, q, then the action of S^1 on $S^2 - \{p,q\} = (-1,1) \times S^1$ is free.

So we set $S^2 = \{p\} \amalg (S^2 - \{p,q\}) \amalg \{q\}$.

We have $(S^2 - \{p,q\})_{S^1} = ((-1,1) \times S^1)_{S^1} = (-1,1) \times (S^1)_{S^1} = (-1,1) \times S^\infty$ (because S^1 acts trivially on $(-1,1)$ and we just proved above that $(S^1)_{S^1} = S^\infty$), has the homotopy type as $(-1,1)$.

Then, $\{p\}_{S^1} = (ES^1 \times \{p\})/S^1 \simeq (ES^1)/S^1 = BS^1 = \mathbb{C}P^\infty$.

Therefore $(S^2)_{S^1} = \{p\}_{S^1} \amalg (S^2 - \{p,q\})_{S^1} \amalg \{q\}_{S^1} = \mathbb{C}P^\infty \amalg (-1,1) \times S^\infty \amalg \mathbb{C}P^\infty$ "a dumbbell") has the same homotopy type as $\mathbb{C}P^\infty \vee \mathbb{C}P^\infty$ ("two $\mathbb{C}P^\infty$'s joining at one point).

[The fiber above p and q is $\mathbb{C}P^\infty$, and the fiber over $(-1,1)$ is S^∞.]

Cohomology of $X := \mathbb{C}P^\infty \amalg (-1,1) \times S^\infty \amalg \mathbb{C}P^\infty$: use the Mayer–Vietoris sequence. Set $U = \mathbb{CP}^\infty \amalg (-1, 1/2)$ and $V = (-1/2, 1) \amalg \mathbb{C}P^\infty$,

We have

$$
\begin{tikzcd}
& X & U \amalg V & U \cap V \\
H^4 & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "i^*"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "j^*"] & \cdots\cdots \\
H^3 & 0 \arrow[r, "i^*"] & 0 \arrow[r, "j^*"] & 0 \arrow[ull, "d^*"] \\
H^2 & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "i^*"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "j^*"] & 0 \arrow[ull, "d^*"] \\
H^1 & 0 \arrow[r, "i^*"] & 0\oplus 0 \arrow[r, "j^*"] & 0 \arrow[ull, "d^*"] \\
H^0 & \mathbb{Z} \arrow[r, "i^*"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "j^*"] & \mathbb{Z} \arrow[ull, "d^*"]
\end{tikzcd}
$$

(from $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[u]$ where u is a degree-2 element.)

So we have

$$
H^k(X) = \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ 0 & \text{for } k \text{ odd}, \\ \mathbb{Z}\oplus\mathbb{Z}, & \text{for } k \text{ even} \end{cases}
$$

Last time we showed that $H^k((S^2)_{S^1}) = \mathbb{Z}[u] \oplus y\mathbb{Z}[u]$, where $\deg(u) = \deg(y) = 2$.

Functoriality:

G-map = G-equivariant map.

A G-map $f\colon N \to M$ of G-spaces induces $f_G\colon N_G \to M_G$.

(Let's show that the map f_G is well defined: $N_G = EG \times N/G$, $M_G = (EG \times M)/G$, by sending $[e, n] \mapsto [e, f(n)]$. But sine $[e, n] = eg, g^{-1}n] \mapsto [eg, f(g^{-1}n)] = [eg, g^{-1}f(n)]$,

Hence, f_G further induces a ring homormophism in cohomology

$$
f_G^*\colon H^*(M_G) \to H^*(N_G)
$$

where

$$
\begin{tikzcd}
N \arrow[r, "f"] \arrow[dr] & M \arrow[d] \\
& pt
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
N_G \arrow[rr, "f_G"] \arrow[dr, "\pi_N"'] & & M_G \arrow[dl, "\pi_M"] \\
& pt_G = BG &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
H_G^*(N) & & H_G^*(M) \ni x \arrow[ll, "f_G^*"'] \\
& H^*(BG) \ni u \arrow[ul, "\beta"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
\tag{1.5}
$$

α and β give the algebra structure, where elements of $H^*(BG)$ serve as scalars:

For any $u \in H^*(BG)$ and $x \in G_H^*(M)$, we have $f_G^*(u \cdot x) = f_G^*(\alpha(u)x) = f_G^*(\alpha(u))f^*(x) = \beta(u)f_G^*(x) = u \cdot f_G^*(x)$. This shows that f_G^* is a $H^*(BG)$-homomorphism, i.e. $H^*(BG)$ is the *scalar* in the algebra.

So: $f_G^*\colon H_B^*(M) \to H_B^*(N)$ is an $H^*(BG)$-algebra homomorphism.

Hence, $H_G^*(-)$ is a contravariant functor from $\{G\text{-spaces}, G\text{-maps}\}$ to $\{H^*(BG)\text{-algebras}, H^*(BG)\text{-homorphisms}\}$.

It is the composition of two functors: $H_G^*(-) = (-)^* \circ (-)_G$.

1.10 Lecture 10: Functoriality

Proposition. Let $f\colon N \to M$ be a G-map of G-spaces.

(i) f injective $\Rightarrow f_G\colon N_G \to M_G$ is injective.

(ii) f surjective $\Rightarrow f_G$ is surjective.

(iii) If $\mathbb{I}\colon M \to M$ is the identity, then $\mathbb{I}_G\colon M_G \to M_G$ is the identity.

(iv) $(h \circ f)_G = h_G \circ f_G$.

(v) If $f\colon N \to M$ is a fiber bundle with fiber F, then $f_G\colon N_G \to M_G$ is also a fiber bundle with fiber F.

(Proof: (i)-(iv) is straightforward so we only show (v): From (1.5), $N_G \to BG$ is a fiber bundle over N; $M_G \to BG$ is a fiber bundle over M. So every point $b \in BG$ has a neighborhood U over which $\pi^{-1}N(U) \simeq U \times N$, $\pi_M^{-1}(U) \simeq U \times M$. $f_G\colon N_G \to M_G$ is locally $U \times N \to U \times M$, which is locally trivial with fiber F.)

Classifying spaces:

Example. $\mathbb{Z}_2$ on S^n by the antipodal map: $S^n \to \mathbb{R}P^n$ is a principal $\mathbb{Z}_2$-bundle,

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d] & S^2 \arrow[r, hook] \arrow[d] & S^3 \arrow[r, hook] \arrow[d] & \cdots \\ \mathbb{R}P^1 \arrow[r, hook] & \mathbb{R}P^2 \arrow[r, hook] & \mathbb{R}P^3 \arrow[r, hook] & \cdots \end{tikzcd}$$

let $S^\infty = \bigcup_{n=1}^\infty S^n$, $\mathbb{R}P^\infty = \bigcup_{n=1}^\infty \mathbb{R}P^n$, there is a $\mathbb{Z}_2$-action on S^∞ with quotient $\mathbb{R}P^\infty$, because S^∞ is contractible, so $S^\infty \to \mathbb{R}P^\infty$ is the universal $\mathbb{Z}_2$-bundle. And $B\mathbb{Z}_2 = \mathbb{R}P^\infty$.

Closed subgroups:

Let $H \subset G$ be a closed subgroup of a topological group G. If G acts freely on EG, then so does H. Let $B = EG/H$.

Proposition: Then $EG \to B$ is locally trivial with fiber H.

(Proof: Since $EG \to BG$ is locally trivial it is locally $U \times G$. So EG/H is locally $(U \times G)/H = U \times (G/H)$, so $EG \to EG/H$ is locally $U \times H \to U \times (G/H)$,

Theorem (Frank Warner's book, *Foundations of Differentiable Manifolds and Lie Groups*). If H is a closed subgroup of Lie group, then $G \to G/H$ is a principle H-bundle.

This means that locally, $G \to G/H$ becomes $V \times H \to V$ for some open set $V \in G/H$.

Therefore $U \times G \to U \times (G/H)$ is locally $U \times V \times H \to U \times V$, so $EG \to EG/H$ is locally trivial with fiber H. We can take $BH = EG/H$.

This implies that if we have a universal bundle for a Lie group, then we have the universal bundle for any of its closed subgroup, i.e. the following theorem

Theorem. If a Lie group G has a universal bundle $EG \to BG$, then any closed subgroup has a universal bundle $EG \to EG/H$.

Theorem. If $\pi_i\colon EG_i \to BG_i$ are universal bundle for $i = 1, 2$, then $\pi_1 \times \pi_2\colon EG \times EG_2 \to BG_1 \times BG_2$, $(e_1, e_2) \mapsto (\pi_1(e_1), \pi_2(e_2))$ is a universal bundle for $G_1 \times G_2$.

(Proof by definition: $(g_1, g_2)(e_1, e_2) = (e_1, e_2) \Leftrightarrow g_1 e_1 = e_1, g_2 e_2 = e_2 \Leftrightarrow g_1 = 1, g_2 = 1$, so $G_1 \times G_2$ acts freely on $EG_1 \times EG_2$. $(\pi_1 \times \pi_2)^{-1}(b_1, b_2) = \{(e_1, e_2) | e_1 \in \pi^{-1}(b_1), e_2 \in \pi^{-1}(b_2)\} = G_1 \times G_2$.)

Corollary: $B(G_1 \times G_2) = BG_1 \times BG_2$. (Here equality is in the sense of up to homotopy equivalence.)

Example. A torus T is $S^1 \times \cdots \times S^1$, therefore $BT = BS^1 \times \cdots \times BS^1 = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$. By the Künneth formula, $H^*(BT) = H^*(BS^1) \otimes \cdots \otimes H^*(BS^1) = \mathbb{Z}[u_1] \otimes \cdots \otimes \mathbb{Z}[u_n] = \mathbb{Z}[u_1, ..., u_n]$ (Since $H^*(\mathbb{C}P^\infty)$ is free abelian).

Well known theorem: Every compact Lie group can be embedded as a closed subgroup of some orthogonal group $O(k)$.

A universal bundle for $O(k)$:

Let $V(k, n) = \{\text{orthonormal } k \text{ frames in } \mathbb{R}^n\} = \{n \times k \text{ matrices} | \text{columns are orthonormal}\}$, can multiply on the right by $A \in O(k)$. A k-frame in $\mathbb{R}^n$ spans a k-plane in $\mathbb{R}^n$, and multiplying on the right by A is just changing the basis. So the quotient $V(k, n)$ by $O(k)$ is the Grassmannian $G(k, n)$.

Fact: $V(i, n) \to G(k, n)$ is a principal $O(k)$-bundle.

Then we have

$$\begin{tikzcd} V(k,n) \arrow[r, hook] \arrow[d] & V(k,n+1) \arrow[r, hook] \arrow[d] & V(k,n+2) \arrow[r, hook] \arrow[d] & \cdots \\ G(k,n) \arrow[r, hook] & G(k,n+1) \arrow[r, hook] & G(k,n+2) \arrow[r, hook] & \cdots \end{tikzcd}$$

Let $V(k, \infty) = \bigcup_{n=k}^\infty V(k, n)$, $G(k, \infty) = \bigcup_{n=k}^\infty G(k, n)$. Then $V(k, \infty)$ is weakly contractble; and in fact it is a CW complex so it is contractible.

Therefore, $V(k, \infty) \to G(k, \infty)$ is the universal $O(k)$-bundle. From this , we arrive at

Theorem. Every compact Lie group G has a universal bundle.

Starting from next time, we will assume all the topological spaces are smooth manifolds and the groups are Lie groups; and we will show equivariant cohomology can be computed using differential forms. This will give us the ring structure of $H^*_{S^1}(S^2)$, which we have not fully determined yet.

1.11 Lecture 11: Review of differential geometry

New chapter today: use differential forms to calculate equivariant cohomology.

de Rham theorem: If M is a C^∞ manifold, and $\Omega^*(M) = \{C^\infty$ differential forms on $M\}$, then $H^*(M;\mathbb{R}) \simeq H\{\Omega^*(M), d\}$, where $H^*(M;\mathbb{R})$ is singular cohomology, and $\Omega^*(M)$ is de Rham complex.

Equivariant de Rham theorem: if a Lie group G acts smoothly on a manifold M, then $H^*_G(M;\mathbb{R}) \simeq H^*\{\Omega^*_G(M), D\}$, where $H^*_G(M;\mathbb{R})$ is singular equivariant cohomology, and $\Omega^*_G(M)$ is called Cartan complex of equivariant differential forms.

Lie derivative of a vector field:

Let $X, Y \in \mathfrak{X}(M) = \{C^\infty$ vector field on $M\}$, $p \in M$, X has an integral curve $\varphi_t(p)$ through p:

$\varphi_t(p) \colon (-\epsilon, \epsilon) \to M$, $\varphi_0(p) = p$, $\frac{d}{dt}\varphi_t(p) = X_{\varphi_t(p)}$,

Actually, $\exists \epsilon > 0$, and a neighborhood U of p in M, s.t. $\varphi \colon (-\epsilon, \epsilon) \times U \to M$ and $\varphi_0(q) = q$, $\forall q \in U$, $\frac{d}{dt}\varphi_t(q) = X_{\varphi_t(q)}$, $\varphi_t \colon U \to \varphi_t(U) \subset M$. And we have $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\varphi_t \colon U \to \varphi_t(U)$, have inverse $\varphi_{-t} \colon \varphi_t(U) \to U$.

Definition (Lie derivative of a vector field). $(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{(\varphi_{-t})_* Y_{\varphi_t(p)} - Y_p}{t} = \frac{d}{dt}(\varphi_{-t})_* Y_{\varphi_t(p)} \in T_p M$.

Lie derivative of a differential form:

Definition (Lie derivative of a differential form). Let $\omega \in \Omega^k(M)$. $(\mathcal{L}_X \omega)_p = \lim_{t\to 0} \frac{\varphi_t^* \omega_{\varphi_t(p)} - \omega_p}{t}$.

Recall if $v_1, ..., v_k \in T_p M$, then $\left(\varphi_t^* \omega_{\varphi_t(p)}\right)(v_1, ..., v_k) := \omega_{\varphi_t(p)}\left(\varphi_{t*} v_1, ..., \varphi_{t*} v_k\right)$.

Definition (Lie derivative of a function). $(\mathcal{L}_X) f = \lim_{t\to 0} \frac{f(\varphi_t(p)) - f(p)}{t} = X_p f$.

Theorem. $\mathcal{L}_X Y = [X, Y]$.

(The proof is a somewhat messy computation.)

Interior multiplication on a vector space: Let V be a vector space. A k-covector on V is an alternating k-linear function on V: $\alpha \colon V \times \cdots \times V \to \mathbb{R}$.

We write $\alpha \in A_k(V) = \Lambda^k(V^\vee)$.

Def. If $v \in V$, then $\iota_v \alpha \in \Lambda^{k-1}(V^\vee)$, $\iota_v \alpha)(v_1, ..., v_{k-1}) = \alpha(v_1, ..., v_{k-1})$. We have $((\iota_v \circ \iota_v)\alpha)(v_1, ..., v_{k-2}) = (\iota_v \alpha)(v, v_1, ..., v_{k-2}) = \alpha(v, v, v_1, ..., v_{k-2}) = 0$.

Definition (Interior multiplication on a manifold). If $X \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$, $Y_1, ..., Y_{k-1} \in \mathfrak{X}(M)$, then $(\iota_X \omega)(Y_1, ..., Y_{k-1}) = \omega(X, Y_1, ..., Y_{k-1})$.

Definition (Derivation, self-defined). A map $D \colon \Omega^*(M) \to \Omega^*(M)$ is a *derivation* if $D(\omega \wedge \tau) = (D\omega) \wedge \tau + \omega \wedge D\tau$

Theorem (Properties of $\mathcal{L}_X$). Theorem (i) $\mathcal{L}_X \colon \Omega^*(M) \to \Omega^*(M)$ is a *derivation* of degree 0. (Derivation means that $\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X \omega) \wedge \tau + \omega \wedge \mathcal{L}_X \tau$).

(ii) $\mathcal{L}_X$ commutes with d: $\mathcal{L}_X \cdot d = d \circ \mathcal{L}_X$.

(iii) (Product formula) if $\omega \in \Omega^k$, $\mathcal{L}_X(\omega(Y_1, ..., Y_k)) = (\mathcal{L}_X \omega)(Y_1, ..., Y_k) + \sum_{i=1}^k \omega(Y_1, ..., \mathcal{L}_X Y_i, ..., Y_k)$.

Theorem (Properties of $\mathcal{L}_X$ and ι_X). (i) $\iota_X \colon \Omega^*(M) \to \Omega^*(M)$ is an antiderivation of degree-1: $\iota_X(\omega \wedge \tau) = (\iota_X \omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \iota_X \tau$,

(ii) $\iota_X \circ \iota_X = 0$

(iii) (Cartan's homotopy formula) $\mathcal{L}_X = d\iota_X + \iota_X d$.

(iv) $\iota_X \colon \Omega^*(M) \to \Omega^*(M)$ is $\mathcal{F}$-linear: $\iota_X(f\omega) = f\iota_X \omega$ for $f \in C^\infty(M) = \mathcal{F}$. (But $\mathcal{L}_X$ is not $\mathcal{F}$ linear, as $\mathcal{L}_X(f\omega) = (\mathcal{L}_X f)\omega + f\mathcal{L}_X \omega$.)

1.12 Lecture 12: Basic forms and invariant forms

Let G be a Lie group, $\pi \colon P \to M$ a C^∞ principal G-bundle.

$\pi \colon P \to M$ is surjective, so $\pi_* \colon T_p P \to T_{\pi(p)} M$ is also surjective, $\pi^* \colon \Omega^*(M) \to \Omega^*$ is unjective.

Definition (Basic form). $\pi^* \Omega^*(M) \subset \Omega^*(P)$ is called the subspace of *basic forms.*

Example: $\pi \colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$

$r \cdot (x, y) = (x, y + r)$, $\omega \in \Omega^1(\mathbb{R}^2)$ is $f(x, y)dx + g(x, y)dy$.

The basic 1-forms are $\pi^*(h(x)dx) = \pi^*(h(x))\pi^*(dx) = \pi^*(h(x))dx$.

$\omega = fdx + gdy$ is basic if and only if $g = 0$ and $f(x, y)$ does not depend on y.

$(i)\iota_{\partial_y} \omega = \iota_{\partial_y}(fdx + gdy) = f\iota_{\partial_y} dx + g\iota_{\partial_y} dy = g$, where we used $\iota_{\partial_y} dx = dx(\partial_y) = \frac{\partial x}{\partial y} = 0$, and $\iota_{\partial_y} dy = 1$.

(ii) $\mathcal{L}_{\partial_y}\omega = \mathcal{L}_{\partial_y}(fdx + gdy) = (\partial_y f)dx + f\mathcal{L}_{\partial_y}dx + (\mathcal{L}_{\partial_y}g)dy + g\mathcal{L}_{\partial_y}(dy) = \frac{\partial f}{\partial y}dx + \frac{\partial g}{\partial y}dy$, where we used $\mathcal{L}_{\partial_y}dx = d(\mathcal{L}_{\partial_y}x) = d(0) = 0$, and $\mathcal{L}_{\partial_y}dy = d(1) = 0$. We see that if $g = 0$ is already guaranteed by $\iota_{\partial_y}\omega = 0$, then $\mathcal{L}_{\partial_y}\omega$ further guarantees that $\partial_y f(x,y) = 0$, i.e. $f(x,y)$ does not depend on y. So we have the following proposition:

Proposition. $\omega = fdx + gdy \in \Omega^1(\mathbb{R}^2)$ is basic for $\pi\colon \mathbb{R}^2 \to \mathbb{R}$, if and only if $\iota_{\partial_y}\omega = 0$ and $\mathcal{L}_{\partial_y}\omega = 0$.

Now, our task is how to generalize this proposition to an arbitrary principal G-bundle.

Vertical vectors on a principal bundle:

Let $\pi\colon P \to M$ be a principal G-bundle, and $p \in P$. Then $\pi_*\colon T_pP \to T_{\pi(p)}(M)$.

Definition (Vertical vector). $\mathfrak{V}_p :=$ {vertical tangent vectors at p} $:= \ker\pi_*$.

An element $A \in \mathfrak{g}$ gives a curve $e^{tA} \in G$. Then e^{tA} defines a curve in M.

Definition ($\underline{A}_p$). If G acts on M smoothly on the left, and $A \in \mathfrak{g}$, the Lie algebra of G, for $p \in M$, define the vector field at p, $\underline{A}_p = \frac{d}{dt}\Big|_{t=0} e^{-tA}\cdot p$. (If G acts on M on the right then the definition changes to $A_p = \frac{d}{dt}\Big|_{t=0} e^{tA}\cdot p$.)

Theorem: $\underline{[A,B]} = [\underline{A},\underline{B}]$ for $A, B \in \mathfrak{g}$. (This sign convention agrees with the sign convention above. Also note that $[A,B]$ is the lie bracket, whereas $[\underline{A},\underline{B}]$ is the commutator for vector fields.)

Theorem. $\underline{A}$ is a C^∞ vector field on M.

Theorem (Integral curve). The *integral curve* of $\underline{A}$ through $p \in M$ is $\varphi_t(p) = e^{-tA}\cdot p$. (Colloquially, "left multiplication by e^{-tA}".)

(Proof. We need to show $\frac{d}{dt}\varphi(t) = \underline{A}_{\varphi_t(p)}$ for all t. ($\frac{d}{dt}\varphi_t(p) = \frac{d}{ds}|_{s=0}e^{-(t+s)A}\cdot p = \underline{A}_{e^{-tA}\cdot p} = \underline{A}_{\varphi_t(p)}$.)

Theorem. $\underline{A}_p$ is a vertical vector.

Fix $p \in P$ and define $j_p\colon G \to P$ by $g \mapsto p\cdot g$. Then $j_{p*}\colon T_eG = \mathfrak{g} \to T_pP$ is given by $j_{p*}(A) = \frac{d}{dt}\Big|_{t=0} j_p(e^{tA}) = \frac{d}{dt}\Big|_{t=0} p\cdot e^{tA} = \underline{A}_p$.

Then $\pi_*(\underline{A}_p) = \pi_* j_{p*}(A) = (\pi\circ j_p)_*(A) = 0$ ($\pi\circ j_p = \pi(p)$, a constant map), so $\underline{A}$ is a vertical vector field.

Invariant forms:

Recall that a basic form on P for a principal bundle $\pi\colon P \to M$ is $\pi^*\omega$ for some $\omega \in \Omega^*(M)$.

We have $r_g^*(\pi^*\omega) = (\pi\circ r_g)^*\omega = \pi^*\omega$.

$((\pi\circ r_g)(p) = \pi(pg) = \pi(p).)$

If G acts on M on the left, each $g \in G$ defines a differential $l_g\colon M \to M$.

Definition (Invariant form). A form $\omega \in \Omega^*(M)$ is *G-invariant* if $l_g^*\omega = \omega$ for all $g \in G$.

So a basic form on P for $\pi\colon P \to M$ is G-invariant.

Theorem (Characterization of invariant forms). Assume G connected, and acts on M. Then $\omega \in \Omega^*(M)$ is G-invariant if and only if $\mathcal{L}_{\underline{A}}\omega = 0$ for all $A \in \mathfrak{g}$, the Lie algebra of G.

Proof. First prove "$\Rightarrow$": If ω is G-invariant, then $l_g^*\omega = \omega$ for all $g \in G$. This implies $\rho_{e^{-itA}}^*\omega = \omega$, for any $A \in \mathfrak{g}$. This is the same as $\varphi_t^*(\omega)$, $\mathcal{L}_{\underline{A}}\omega = \frac{d}{dt}\Big|_{t=0}\varphi_t^*\omega = \frac{d}{dt}\Big|_{t=0}\omega = 0$.

The "$\Leftarrow$" part of the prove will be given in the next lecture.

1.13 Lecture 13: Basic forms

Today we want to characterize basic forms using differential forms.

The homotopy quotient M_G is the base space of a principal G-bundle $EG \times M \to M_G$. Let G be a Lie group. We work in the C^∞ category.

Definition of invariant forms – see last lecture.

Continue the proof in the last lecture: $\Leftarrow$: suppose $\mathcal{L}_{\underline{A}}\omega = 0$.

Let $p \in M$. An integral curve of $\underline{A}$ through p is $\varphi_t(p) = e^{-tA}\cdot p = l_{e^{-tA}}(p)$.

We have $\mathcal{L}_{\underline{A}}\omega = 0 \Rightarrow (\mathcal{L}_{\underline{A}}\omega)_p = \frac{d}{dt}\Big|_{t=0}(\varphi_t^*\omega)_p = \frac{d}{dt}(l_{e^{-tA}}^*\omega)_p = 0$, define $h(t) := (l_{e^{-tA}}^*\omega)_p\colon \mathbb{R} \to \Lambda^k(T_p^*(M))$ is constant. (Here we have assumed that ω is a k-form.) We want to show $h(t) = h(0) = \omega_p$ is constant.

$h'(t) = \frac{d}{ds}\big|_{s=0} l_{e^{-(t+s)A}}^*\omega = \frac{d}{ds}\big|_{s=0} l_{e^{-tA}}^*\left(l_{e^{-sA}}^*\omega\right)_{e^{-tA}\cdot p} = l_{e^{-tA}}^*\frac{d}{ds}\big|_{s=0}\left(l_{e^{-sA}}^*\omega\right)_{e^{-tA}\cdot p}$ (the pullback of differential forms commutes with $\frac{d}{ds}$ because $l_{e^{-tA}}^*$ is linear.) But we have $\frac{d}{ds}\big|_{s=0}\left(l_{e^{-sA}}^*\omega\right)_{e^{-tA}\cdot p} = (\mathcal{L}_{\underline{A}}\omega)_{e^{-tA}p} = 0$, so $h'(t) = 0$, $h(t) = \omega_p$ is constant. Since G is connected, G is generated by any neighborhood of the identity. $\exists$ a neighborhood U of the identity s.t. $e^(-)\colon \mathfrak{g} \to G$ is a diffeomorphism on U, so every $g \in G$ is a product of finitely many exponentials

Thus $(l_g^*\omega)_p = \omega_p$ for all $g \in G$, $p \in M$.

Vertical vectors: let $\pi\colon E \to M$ be a fiber bundle with fiber F. $p \in E$, then $\pi_*\colon T_pE \to T_{\pi(p)}M$ is surjective.

In last lecture we defined the set of vertical vectors at p to be $\ker\pi_* := \mathfrak{V}_p$.

We have also defined j_p and j_{p*}, which acting on A gives the fundamental vector field: $(j_p)_*(A) = \underline{A}_p \in \mathfrak{V}_p$.

This defines a map $(j_p)_*\colon \mathfrak{g} \to \mathfrak{V}_p$.

Lemma. Let $A \in \mathfrak{g}$. Then $\underline{A}_p = 0$ if and only if p is a fixed point of the curve $\{e^{tA} \in G\}$.

Proof: "$\Leftarrow$": $\underline{A}_p = \frac{d}{dt}\big|_{t=0} p\cdot e^{tA} = \frac{d}{dt}\big|_{t=0} p = 0$ (if p is a fixed point of e^{tA}. "$\Rightarrow$": suppose $\underline{A}_p = 0$, an integral curve of $\underline{A}$ through p is $\varphi_t(p) = p\cdot e^{tA}$. Let $c(t) = p$. Then $c'(t) = 0 = \underline{A}_p = \underline{A}_{c(t)}$, so that $c(t)$ is another integral curve of $\underline{A}$ through p. By the uniqueness of integral curves, $\varphi_t(p) = p$ for all $t \in \mathbb{R}$, i.e. $p\cdot e^{tA} = p$ for all $t \in \mathbb{R}$. If $(j_p)_*(A) = \underline{A}_p = 0$, then p is a fixed point of e^{tA}, so $\mathrm{Stab}(p) \supset \{e^{tA} | t \in \mathbb{R}\}$. Since P is a principal bundle, G acts freely on P, so $\mathrm{Stab}(p) = \{1\}$, so $\{e^{tA} | t \in \mathbb{R}\} = \{1\}$, hence $A = 0$. So $(j_p)_*\colon \mathfrak{g} \to \mathcal{V}_p$ is injective.

Since $\dim\mathfrak{g} = \dim G = \mathfrak{V}_p$, $(j_p)_*$ is an isomorphism.

Horizontal forms:

Let $\pi\colon E \to M$ be a fiber bundle.

Definition (Horizontal form). A form $\omega \in \Omega^*(E)$ is *horizontal* if at any $p \in E$, $\iota_{Y_p}\omega = 0$, $\forall Y_p \in \mathfrak{V}_p$.

Basic forms are horizontal: if $\omega = \pi^*(\tau) \in \Omega^k(E)$ and $v_1, ..., v_{k-1} \in T_p(E)$, then $(\iota_{Y_p}\omega)(v_1, ..., v_{k-1}) = \omega(Y_p, v_1, ..., v_{k-1}) = (\pi^*T)_p(Y_p, ...) = T_{\pi(p)}(\pi_*Y_p, ...) = T_{\pi(p)}(0, ...) = 0$.

1.14 Lecture 14: Basic forms, ring structure on $H^*(E)$.

Characterization of basic forms:

Theorem (Basic $\Leftrightarrow$ invariant + horizontal). Let G be a connected Lie group, and $\pi\colon P \to M$ a principal G-bundle. A form $\omega \in \Omega^k(P)$ is basic if and only if it is horizontal, i. $\iota_{\underline{A}}\omega = 0$, and $\mathcal{L}_{\underline{A}}\omega = 0$ for all $A \in \mathfrak{g}$.

Proof: "$\Rightarrow$" has been done in the last lecture. Now, "$\Leftarrow$": Suppose $\iota_{\underline{A}}\omega = 0$, and $\mathcal{L}_{\underline{A}}\omega = 0$ for all $A \in \mathfrak{g}$, since G is connected, ω is G-invariant, let $m \in M$, $w_1, ..., w_k \in T_mM$, pick any $p \in \pi^{-1}(m) \subset P$, and $v_1, ..., v_k \in T_pP$, s.t. $\pi_*v_i = w_i$, for all i. Then define $\tau_m(w_1, ..., w_k) = \omega_p(v_1, ..., v_k)$. Then $\omega = \pi^*\tau$, because

$$\tau_m(w_1, ..., w_k) = \tau_m(\pi_*v_1, ..., \pi_*v_k) = (\pi^*\tau)_p(v_1, ..., v_k), \tag{1.6}$$

therefore we have found the form downstars on M. To show that it is a basic form, we still need to show that the form is well-defined, i.e. we need to show that τ is independent of the choice of v_i and p:

prove independence of v_i: suppose $v_1' \in T_pP$ is another vector s.t. $\pi_*v_1' = w_1 = \pi_*v_1$, then $\pi_*(v_1' - v_1) = 0$, so $v_1' - v_1$ is vertical, so $v_1' - v_1 = \underline{A}_p$ for some $A \in \mathfrak{g}$. Since $\iota_{\underline{A}}\omega = 0$, $0 = \omega_p(\underline{A}_p, v_2, ..., v_k) = \omega_p(v_1', v_1, v_2, ..., v_k)$, so $\omega_p(v_1', v_2, ...) = \omega_p(v_1, v_2, ...)$, showing that the definition (1.6) is independent of the choice of v_1, and simiarly independent of the choice of $v_2, v_3, \ldots$.

Then prove independence of point: suppose $p' \in \pi^{-1}(m)$ is another point in P above m. Then because G acts freely and transitively, so $p' = pg$ for some $g \in G$. Let $v_1', ..., v_k' \in T_{p'}P$ s.t. $\pi_*v_i' = w_i$, then $\omega_{p'}(v_1', ..., v_k') = \omega_{pg}(v_1', ..., v_k') = (r_{g^{-1}}^*\omega)_{pg}(v_1', ..., v_k')$ (because ω is G-invariant), so $\omega_{p'}(v_1', ..., v_k') = r_{g^{-1}}^*(\omega_{pgg^{-1}})(v_1', ..., v_k') = \omega_p(r_{g^{-1}*}v_1', ..., r_{g^{-1}*}v_k')$, since $\pi_*r_{g^{-1}*}v_i' = \pi_*v_i' = w_i$, so $\omega_{p'}(v_1', ..., v_k') = \omega_p(v_1, ..., v_k)$. This shows that the definition (1.6) is independent of the choice of $p \in \pi^{-1}(m)$.

Thus τ is well-defined, and $\omega = \pi^*\tau$ is basic.

[Summary of lecture 12-14: an element is basic (meaning that it comes from the base) iff it is horizontal and invariant. If the group G is connected, it is invariant if and only if its Lie derivative is zero with respect to all vertical vector fields.]

Ring structure on $H^*(E)$:

Product structure on associated graded module: Let $\pi\colon E \to B$ be a fiber bundle with fiber F. Let $H^*(-)$ be cohomology with coefficients in any commutative ring R with identity 1.

Leray's theorem: There is a filtration $H^*(E) = F_0 \supset F_1 \supset F_2 \supset \cdots$ so that multiplication in $H^*(E)$ induces a map $F_k \times F_l \to F_{k+l}$.

It follows that $F_k \times F_{l+1} \to F_{k+l+1}$, and $F_{k+1} \times F_l \to F_{k+l+1}$, therefore there is an induced map

$$\frac{F_k}{F_{k+1}} \times \frac{F_l}{F_{l+1}} \to \frac{F_{k+l}}{F_{k+l+1}},$$

This is the product structure on $GH^*(E) = \bigoplus_{k=0}^{\infty} \frac{F_k}{F_{k+1}}$.

Theorem (Spectral sequence of a filtered complex). $E_\infty \simeq GH^*(E)$ as isomorphic rings.

See Lecture 22 and 23 of the Bott-Tu notes.

Example: Cohomology ring of $U(2)$.

$U(2)$ acts on $\mathbb{C}^2$: it preserves the unit sphere $S^3 \subset \mathbb{C}^2$.

$U(2)$ acts transitively on S^3; $\mathrm{Stab}((1,0)) \simeq S^1$. By the orbit-stabilizer group, $U(2)/U(1) \simeq \mathrm{Orbit}((1,0)) = S^3$ (as $U(2)$ acts on S^3 transitively). Since $U(1)$ is a closed subgroup of the Lie group $U(2)$, there is a fiber bundle

$$\begin{array}{ccc} U(1) & \longrightarrow & U(2) \\ & & \downarrow \\ & & S^3 \end{array}.$$

Since S^3 is simply connected, we have

$$E_2 = H^*(S^3) \otimes H^*(S^1),$$

we have the spectral sequence

$$E_2 = \begin{array}{r|llllll} q=2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q=1 & \bar{x} & 0 & 0 & \bar{x}\bar{y} & 0 & \cdots \\ q=0 & 1 & 0 & 0 & \bar{y} & 0 & \cdots \\ \hline & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots \end{array},$$

where all the $\cdots$ are zero entries. All the differentials d_2 are forced to zero. Using $E_2 = \cdots = E_\infty = GH^*(U(2))$.

$H^0(U(2)) = \mathbb{Z} \cdot 1$,

$H^1(U(2)) = F_0^1 \supset F_1^1 \supset F_2^1 = 0$, where $F_0^1/F_1^1 = \mathbb{Z}x$, $F_1^1/F_2^1 = 0$, so $F_1^1 = 0$, and $H^1(U(2)) = F_0^1 = \mathbb{Z}\bar{x}$. (since $\bar{x}$ is actually in $H^1(U(2))$, we will write $x = \bar{x}$.)

Similarly, $\bar{y} \in H^3(U(2))$, and $\bar{x}\bar{y} \in H^4(U(2))$, so we can write $y = \bar{y}$, and $xy = \bar{x}\bar{y}$. So we get

$$H^k(U(2)) = \begin{cases} \mathbb{R} \cdot 1 & k = 0 \\ \mathbb{R} \cdot x & k = 1 \\ \mathbb{R} \cdot y & k = 3 \\ \mathbb{R} \cdot xy & k = 4 \\ 0 & \text{otherwise} \end{cases} = \mathbb{R}(x,y)/(x^2, y^2, xy + yx) = \Lambda(x_1, x_2),$$

where $\mathbb{R}(x,y)$ is the free algebra generated by x, y, and $\Lambda(x_1, x_2)$ is the *free exterior algebra* defined as follows: $\Lambda(x_1, ..., x_k) = \frac{\mathbb{R}(x_1, ..., x_k)}{(x_i x_j - (-1)^{\deg x_i \cdot \deg x_j} x_j x_i)}$.

1.15 Lecture 15: Vector-valued forms

Let's assume all vector spaces are finite dimensional and are over $\mathbb{R}$.

A *k-covector* on a vector space T is a k-linear alternating function $T^k = \underbrace{T \times T}_{k \text{ times}} \to \mathbb{R}$.

Let V be a vector space. A V-valued k-covector on T is a k-linear alternating function $f\colon \underbrace{T \times T}_{k \text{ times}} \to V$.

By the universal orpoerty of $\Lambda^k T$,

$$\begin{tikzcd} \Lambda^k T \arrow[dr, dotted, "\exists! \text{ linear } \tilde{f}"] & \\ T^k \arrow[u] \arrow[r, "f"] & V \end{tikzcd}$$

Notation: $A_k(T, V) = \{V\text{-valued } k\text{-covectors}\} \simeq \mathrm{Hom}(\Lambda^k T, V) \simeq (\Lambda^k T)^\vee \otimes V \simeq (\Lambda^k T^\vee) \otimes V$, where the last two follows from linear algebra.

Def. A V-valued k-form on a manifold M is a function that assigns to each $p \in M$ a V-valued k-covector in $(\Lambda^k T_p^* M) \otimes V$, i.e. it is a section of the bundle $(\Lambda^k T^* M) \otimes V$.

Notation. $\Omega^k(M; V) = \{C^\infty\ V\text{-valued } k\text{-forms}\}$. If $e_1, ..., e_n$ is a basis for V, and $\omega \in \Omega^k(M; V)$, $v_1, ..., v_k \in T_p M$, then $\omega_p(v_1, ..., v_k) = \sum_{i=1}^n \omega_p^i(v_1, ..., v_k) e_i$, i.e. $\omega = \sum \omega^i e_i$, where ω^i are $\mathbb{R}$-valued k-forms on M.

Def. $d\omega = \sum (d\omega^i) e_i$ if $\omega = \sum \omega^i e_i$.

(This definition is independent of the basis $e_1, ..., e_n$.)

$\mathfrak{g}$-valued forms:

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\omega \in \Omega^k(M; \mathfrak{g})$ and $\tau \in \Omega^l(M, \mathfrak{k})$.

Def. $[\omega,\tau](v_1,...,v_k,v_{k+l}) = \sum_{(k,l)\text{-shuffles } \sigma\in S_{k+l}} \operatorname{sgn}(\sigma)\left[\omega_p(v_{\sigma(1),...,\sigma(k)})\tau_p(v_{\sigma(k+1)},...,v_{\sigma(k+l)})\right]$, here (k,l)-shuffles denote the σ satisfying $(\sigma(1) < \cdots \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)$.

Example. If $\omega,\tau \in \Omega^1(M,\mathfrak{g})$, then $[\omega,\tau](X,Y) = [\omega(X),\tau(Y)] - [\omega(Y),\tau(X)]$.

Proposition. If $X_1,...,X_n$ is a basis for the Lie algebra $\mathfrak{g}$, and $\omega = \sum \omega^i X_i$, $\tau = \sum \tau^j X_j$, then (i) $[\omega,\tau] = \sum \omega^i \wedge \tau^j [X_i,X_j]$.

(ii) $[\tau,\omega] = (-1)^{(\deg\omega)(\deg\tau)+1}[\omega,\tau]$ (note the extra plus one in the sign). (iii) $d[\omega,\tau] = [d\omega,\tau] + (-1)^{\deg\omega}[\omega,d\tau]$.

$\mathfrak{gl}(n,\mathbb{R})$-valued forms:

$\mathfrak{gl}(n,\mathbb{R})$ has two multiplications: $[-,-]$ and matrix product.

A basis for $\mathfrak{gl}(n,\mathbb{R})$ is $\{e_{ij}\}_{1\le i,j\le n}$, where e_{ij} is the $n\times n$ matrix with 1 in the (i,j) position, and zero anywhere else.

We have $e_{ij}e_{hl} = \delta_{jk}e_{il}$,

Def. If $\omega = \sum \omega^{ij}e_{ij}$ and $\tau = \sum \tau^{kl}e_{kl}$, then we define $\omega\wedge\tau = \sum \omega^{ij}\wedge\tau^{kl}e_{ij}e_{kl} = \sum \omega^{ik}\wedge\tau^{kl}e_{il}$.

Proposition. If $\omega,\tau \in \Omega^*(M,\mathfrak{gl}(n,\mathbb{R}))$, then $[\omega,\tau] = \omega\wedge\tau - (-1)^{\deg\omega\deg\tau}\tau\wedge\omega$ (note that the sign does not contain extra minus one).

Corollary: $[\omega,\omega] = \begin{cases} 0 & \text{if } \deg\omega \text{ is even,} \\ 2\omega\wedge\omega & \text{if } \deg\omega \text{ is odd.} \end{cases}$

Fundamental vector field:

Suppose a Lie group G acts on a manifold P on the right.

Proposition. For $A \in \mathfrak{g} = \mathrm{lie}(G)$, $p \in P$, $g \in G$,

$$r_{g*}(\underline{A}_p) = \underline{(\mathrm{Ad}g^{-1})A}_{pg}.$$

Let $c(g)\colon G \to G$ be conjugation by g, $c(g)(x) = gxg^{-1}$.

Def. The differential $\mathrm{Ad}(g) = c(g)_*\colon T_eG \to T_e(G)$.

Proof: Let $j_p\colon G \to P$ be $j_p(g) = p\cdot g$. Then $(j_p)_*(A) = \frac{d}{dt}\Big|_{t=0} j_p(e^tA) = \frac{d}{dt}\Big|_{t=0} p\cdot e^{tA} = \underline{A}_p$.

$(r_g\circ j_p)(x) = pxg = pg\cdot(g^{-1}xg) = pg\cdot c(g^{-1}(x) = j_{pg}(c(g^{-1})(x)) = \left(j_{pg}\circ c(g^{-1})\right)(x)$, so $r_g\circ j_p = j_{pg}\circ c(g^{-1})$.

By chain rule, $(r_g)_*(\underline{A}_p) = (r_g)_*(j_p)_*(A) = (j_{pg})_*(c(g^{-1}))_*(A) = (j_{pg})_*((\mathrm{Ad}g^{-1})A) = \underline{(Adg^{-1})A}_{pg}$.

1.16 Lecture 16

Connection on a principal bundle

1.17 Lecture 17: Curvature on a principal bundle

Review of last lecture: let G be a Lie group, $\pi\colon P \to M$ a principal G-bundle. A *connection* on $P \to M$ is a C^∞ right-invariant horizontal distribution, or, equivalently, it is a $\mathfrak{g}$-valued 1-form, ω, on P, s.t. (i) for $A \in \mathfrak{g}$, $\omega_p(\underline{A}_p) = A$, and (ii) $r_g^*(\omega) = (\mathrm{Ad}g^{-1})\omega$.

The *Maurer–Cartan form* on G is the unique left-invariant $\mathfrak{g}$-valued 1-form θ on G s.t. $\theta_e(X_e) = X_e$ for $X_e \in \mathfrak{g} = T_eG$.

θ satisfies the *Maurer–Cartan equation*: $d\theta + \frac{1}{2}[\theta,\theta] = 0$.

Example: $\begin{array}{ccc} M\times G & \xrightarrow{\pi_2} & G \\ \Big\downarrow{\scriptstyle \pi_1} & & \\ M & & \end{array}$. Let $\omega = \pi_2^*\theta$, a $\mathfrak{g}$-valued 1-form on $M\times G$. Claim: ω is a connection on $M\times G \to M$.

We have $d\omega + \frac{1}{2}[\omega,\omega] = 0$,

Definition (Curvature of a principal bundle). The $\mathfrak{g}$-valued 2-form $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ on a principal bundle P is called the *curvature* of the connection ω.

It is a measure of the deviation of ω from the Maurer–Cartan connection.

With respect to a basis $X_1,...,X_n$ of $\mathfrak{g}$, $\omega = \sum \omega^k X_k$, $\Omega = \sum \Omega^k X_k$, where ω^k and Ω^k are $\mathbb{R}$-valued forms on P.

We have $\sum \Omega^k X_k = \sum(d\omega^k)X_k + \frac{1}{2}[\sum \omega^i x_i, \omega^j x_j] = \sum(d\omega^k)X_k + \frac{1}{2}\sum \omega^i\wedge\omega^j c_{ij}^k x_k$, so

Theorem (Second structural equation).

$$\Omega^k = d\omega^k + \frac{1}{2}\sum_{ij} c_{ij}^k \omega^i\wedge\omega^j.$$

(The 1st structural equation is for the principal bundle associated with the tangent bundle.)

Theorem (Bianchi's second identity).

$$d\Omega = [\Omega,\omega].$$

(Uses $[d\omega, \omega] = -[\omega, d\omega]$, and $[[\omega, \omega], \omega] = 0$.)

Theorem. The curvature Ω on P satisfies (i) Ω is horizontal, i.e. $\iota_{X_p}\Omega = 0$ for any vertical vector $X_p = \underline{A}_p$ for some $A \in \mathfrak{g}$. (ii) $r_g^*\Omega = (\mathrm{Ad}g^{-1})\Omega$, (iii) $\Omega_p(X_p, Y_p) = (d\omega)_p(hX_p, hY_p)$. where $X_p = vX_p + hX_p$ is the decomposition of X_p into vertical and horizontal components.

(Proof can be found in Tu's book on Differential geometry.)

Suppose a Lie group G acts on a C^∞ manifold M. Then $\Omega(M) = \{C^\infty \text{ forms on } M\}$ is a differential graded algebra (dga).

On $\Omega(M)$, we can define 2 actions of $\mathfrak{g}$: (i) $\iota_A\tau := \tau_{\underline{A}}\tau$, and (ii)$\mathcal{L}_A\tau := \mathcal{L}_{\underline{A}}\tau$.

We can also define an action of G: for $g \in G$, (iii) $g \cdot \tau = r_g^*\tau$, Cartan homotopy formula $\mathcal{L}_A = d\iota_A + \iota_A d$.

Def A dga with athe action (i), (ii), (iii) satisfying (iv) is called a G-dga.

Example: if M is a G-manifold, then $\Omega(M)$ is a G-dga.

The Weil algebra is

$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$.

Definition (The Weil map). The *Weil map* $f\colon W(\mathfrak{g}) \to \Omega(P)$ is defined as follows. Let $\alpha \in \mathfrak{g}^\vee$, $p \in P$, then

$$T_pP \xrightarrow{\omega_p} \mathfrak{g} \xrightarrow{\alpha} \mathbb{R},$$

then $\alpha \circ \omega_p$ is an $\mathbb{R}$-valued 1-form on P, as p varies over P.

This gives a map $f \circ \mathfrak{g}^\vee \to \Omega^1(P)$, $f(\alpha) = \alpha \circ \omega$.

We can extend $f\colon \Lambda(\mathfrak{g}^\vee) \to \Omega(P)$ as an algebra homomorphism, i.e. if $X_1, ..., X_n$ is a basis of $\mathfrak{g}$, $\alpha^1, ..., \alpha^n$ is the dual basis, then $\Lambda(\mathfrak{g}^\vee) = \Lambda(\alpha^1, ..., \alpha^n)$, we define $f(\alpha^1 \wedge \cdots \wedge \alpha^n) = f(\alpha^1) \wedge \cdots \wedge f(\alpha^n)$.

This gives the Weil map $f\colon \Lambda(\mathfrak{g}^\vee) \to \Omega(P)$.

1.18 Lecture 18: The Weil algebra

Let $P \to M$ be a principal G-bundle with a connection ω.

where G is a Lie group with Lie algebra $\mathfrak{g}$.

Let $X_1, ..., X_n$ be a basis for $\mathfrak{g}$, and $\alpha^1, ..., \alpha^n$ the dual basis for $\mathfrak{g}^\vee$.

Definition (Weil algebra). The *Weil algebra*

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) = \Lambda(\alpha^1, ..., \alpha^n) \otimes \mathbb{R}[\alpha^1, ..., \alpha^n],$$

where in $\Lambda(\alpha^1, ..., \alpha^n)$, $\alpha^i \wedge \alpha^j = -\alpha^j \wedge \alpha^i$, and in $\mathbb{R}[\alpha^1, ..., \alpha^n]$, $\alpha^i\alpha^j = \alpha^j\alpha^i$.

Let $\theta_i = \alpha^i \otimes 1 \in W(\mathfrak{g})$, $u_0 = 1 \otimes \alpha^i \in W(\mathfrak{g})$. Then we have $W(\mathfrak{g}) = \Lambda(\theta_i, ..., \theta_n) \otimes \mathbb{R}[u_1, ..., u_n]$.

We give a grading by $\deg\theta_i = 1$, and $\deg u_i = 2$.

We defined the Weil map $f\colon \Lambda(g^\vee) \to \Omega(P)$.

Similarly, if Ω is the curvature of ω and $p \in P$, $\alpha \in \mathfrak{g}^*$, then

$$T_pP \xrightarrow{\omega_p} \mathfrak{g} \xrightarrow{\alpha} \mathbb{R}$$

gives an $\mathbb{R}$-valued 2-form $\alpha \circ \Omega_p$ in P.

Then there is a map $f\colon \mathfrak{g}^\vee \to \Omega^2(P)$.

We can extend f to an algebra homomorphism $f\colon S(\mathfrak{g}^\vee) \to \Omega(P)$,

where $S(\mathfrak{g}^\vee) = \mathbb{R}[u^1, ..., u^n]$ with $f(u^{ij}, ..., u^{kl}) = f(u^{ij}) \wedge \cdot \wedge f(u^{kl})$.

So we have a bilinear map $\widetilde{f}\colon W(\mathfrak{g}) \to \Omega(P)$,

$(\alpha, \beta) \mapsto f(\alpha) \wedge f(\beta)$.

hence, there is a lienar map $f\colon W(\mathfrak{g}) \to \Omega(P)$, where $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^\vee) \otimes_{\mathbb{R}} S(\mathfrak{g}^\vee)$. This map is called the *Weil map*.

We want f to be a morphism of G-dga.

$$\begin{array}{ccc} W(\mathfrak{g}) & \xrightarrow{f} & \Omega(P) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ W(\mathfrak{g}) & \xrightarrow[f]{} & \Omega(P) \end{array}$$

with respect to a basis $X_1, ..., X_n$ for $\mathfrak{g}$ and dual basis $\alpha^1, ..., \alpha^n$ for $\mathfrak{g}^\vee$,

$\omega = \sum \omega^i X_i$, $\Omega = \sum \Omega^i X_i$.

$f(\theta^i) = \theta^i \circ \omega = \theta^i \cdot (\sum \omega^i X_i) = \sum_i \omega^i \theta^k(X_i) = \omega^k$, and $f(u^k) = u^k \cdot \Omega = u^k \circ (\sum \Omega^i X_i) = \Omega^k$.

How should $d\theta^k$ be defined in $W(\mathfrak{g})$?

By the second structural equation, $\Omega^k = d\omega^k + \frac{1}{2}\sum c_{ij}^k \omega^i \wedge \omega^j$.

For f to preserve d, we must define $d\theta^k = u^k - \frac{1}{2}\sum c_{ij}^k \theta^i \wedge \theta^j$.

By Bianchi's second identity, $d\Omega^k = \sum c_{ij}^k \Omega^i \wedge \omega^j$.

So we must define $du^k = \sum c_{ij}^k u^i \theta^j$.

We can extend d to the Weil algebra $W(\mathfrak{g}) \to W(\mathfrak{g})$ as an antiderivation of degree 1.

This makes $W(\mathfrak{g})$ into a dga.

Now define the action of Lie aigebra on the Weil algebra:

Let $A \in \mathfrak{g}$, we have

$$\begin{tikzcd} \theta^k \arrow[r] \arrow[d] & \omega^k \arrow[d] \\ \iota_A(\theta^k) \arrow[r] & \iota_A(\omega^k) = \tau_{\underline{A}}\omega^k = \omega^k(\underline{A}) \end{tikzcd}$$

Since $\omega(\underline{A}) = A = \sum \theta^k(A)x_k$, so $\omega^k(\underline{A}) = \text{const}\theta^k(A)$, so we must define $\iota_A(\theta^k) = \theta^k(A)$, in order to make the diagram above commute.

$$\begin{tikzcd} u^k \arrow[r] \arrow[d, "\iota_A"'] & \Omega^k \arrow[d, "\iota_A"] \\ \iota_A u^k \arrow[r] & \iota_A(\Omega^k) = \tau_{\underline{A}}\Omega^k = 0 \end{tikzcd}$$

because Ω is horizontal.

So we must define $\iota_A u^k = 0$.

We can extend ι_A to $W(\mathfrak{G})$ as an antederivation of deg -1.

Finally, we define $\mathcal{L}_A = d\iota_A + \iota_A d$. Because both d and τ_A commute with f, $\mathcal{L}_A$ will also.

If $g \in G$,

$$\begin{tikzcd} \theta^k \arrow[r, "f"] \arrow[d, "r_g^*"'] & \omega^k \arrow[d, "r_g^*"] \\ r_g^*\theta^k \arrow[r] & \left[(\mathrm{Ad}g^{-1})\omega\right]^k \end{tikzcd}$$

Let $\theta = \sum \theta^k X_k$, $u = \sum u^k X_k$. We define $r_g^*\theta = (\mathrm{Ad}g^{-1})\theta$, so $r_g^*\theta^k = \left[(\mathrm{Ad}g^{-1})\theta\right]^k$, and $r_g^*\theta^k = \left[(\mathrm{Ad}g^{-1})u\right]^k$.

Extend r_g^* to $W(\mathfrak{g}) \to W(\mathfrak{g})$ as an algebra homomorphism. This makes $W(\mathfrak{g})$ into a G-dga.

1.19 Lecture 19: The Weil algebra

Let G be a Lie group with Lie algebra $\mathfrak{g}$.

Assume G connected.

The a G-dga has 2 operations: ι_A, $\mathcal{L}_A$, in addition to those of a dga.

(We do not need r_g^*.)

The Weil algebra of G is, given a basis $X_1, ..., X_n$ for $\mathfrak{g}$, $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) = \Lambda(\theta_1, ..., \theta_n) \otimes \mathbb{R}[u_1, ..., u_n]$.

$d\theta_k = u_k - \frac{1}{2}\sum c_{ij}^k \theta_i\theta_j$, $du_k = \sum c_{ij}^k u_i\theta_j$, $\iota_A\theta_k = \theta_k(A)$, $\iota_A u_k = 0$, $\mathcal{L}_A\theta_k = d\iota_A\theta_k + \tau_A d\theta_k = 0 - \frac{1}{2}\iota_A(\sum c_{ij}^k \theta_i\theta_j)$, $\mathcal{L}_A u_k = d\iota_A u_k + \tau_u du_k = \iota_A du_k$.

Extend d to $W(\mathfrak{g})$ as antiderivation

Proposition: $d^2 = 0$ on $W(\mathfrak{g})$

(Since d is an derivation, d^2 is a derivation.)

Proof: it is enough to check $d^2 = 0$ on a set of algebra generators of $W(\mathfrak{g})$. I.e. $\theta_1, ..., \theta_n, u_1, ..., u_n$, or $\theta_1, ..., \theta_n, d\theta_1, ..., d\theta_k$.

We have $d\theta = u - \frac{1}{2}[\theta, \theta]$. Then one can check that $d^2\theta = 0$. This says $d^2\theta^k = 0$ for all k. This shows that $d^2 = 0$ on a set of generators of $W(\mathfrak{g})$. So $d^2 = 0$ on $W(\mathfrak{g})$.

Theorem. $H^*(W(\mathfrak{g}), d) = \begin{cases} \mathbb{R} & \text{in deg } 0 \\ 0 & \text{in deg} > 0. \end{cases}$

Proof. It is enough to find a cochain homotopy $K\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ of degree -1 s.t. $dK + Kd = 1$ (so that 1 is homotopic to 0).

Recall $d\theta_k = u_k - \frac{1}{2}\sum c_{ij}^k \theta_i\theta_j := z_k$, $du_k = \sum c_{ij}^k u_i\theta_j$.

Then $\theta_i, ..., \theta_n, z_1, ..., z_n$ is a set of generators for $W(\mathfrak{g})$.

Define $\bar{K}\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ by $\bar{K}\theta_k = 0$, $\bar{K}z_k = \theta_k$, then it's easy to check $(d\bar{K} + \bar{K}d)\theta_k = \theta_k$ and $(d\bar{K} + \bar{K}d)z_k = z_k$. But $(d\bar{K} + \bar{K}d)(\theta_i z_j) = 2\theta_i z_j$. To remedy this, we define $K = \frac{1}{p+q}\bar{K}$ on $\Lambda^p(\mathfrak{g}^\vee) \otimes S^q(\mathfrak{g}^\vee)$ for $(p, q) \neq (0, 0)$. This gives $dK + Kd = 1$ on $W(\mathfrak{g})$ except in degree 0. This shows that $H^*(W(\mathfrak{g}, d) = 0$ in degree > 0. In degree 0, $H^0(W(\mathfrak{g})) = \mathbb{R}$ because $d(1) = 0$.

[Below is from a correction in lecture 21:]

Cohomology of the Weil algebra: Let G be a Lie group with Lie algebra $\mathfrak{g}$. $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$. If $X_1, ..., X_n$ is a basis for $\mathfrak{g}$ and $\theta_1, ..., \theta_n$ is the dual basis for $\mathfrak{g}^\vee$ in $\Lambda(\mathfrak{G}^\vee)$, and $u_1, ..., u_n$ is the dual basis for $\mathfrak{g}^\vee$ in $S(\mathfrak{G}^\vee)$, then $W(\mathfrak{g}) = \Lambda(\theta_1, ..., \theta_n) \otimes \mathbb{R}[u_1, ..., u_n]$.

Let $z_k = d\theta_k = u_k - \frac{1}{2}\sum_{i,j} c^k_{i,j}\theta_i \wedge \theta_k$, $du_k = \sum c^k_{ij} u_i \theta_j$. Then $d\theta_k = z_k$, $dz_k = 0$.

We defined an antiderivation $\bar{K}\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ with $\bar{K}z_k = \theta_k$,, $\bar{K}\theta_k = 0$. We found $d\bar{K} + \bar{K}d = 1$ on θ_k, z_k. We defined $K = \frac{1}{p+q}\bar{K}$ on $\bigoplus_{p+q>0} \Lambda^p(\theta_1, ..., \theta_n) \otimes S^q(z_1, ..., z_n)$. Then $dK + Kd = 1$.

Note that K is not an antiderivation. If $\alpha \in \Lambda^p(\theta_1, ..., \theta_n) \otimes S^q(z_1, ..., z_n)$, then deg$\alpha = p + 2q$. Then $H^*(W(\mathfrak{g})) = \begin{cases} 0, & \deg > 0 \\ \mathbb{R}, & \deg = 0 \end{cases}$

1.20 Lecture 20: The Weil & Cartan model

Since the Weil algebra $W(\mathfrak{g})$ has the same cohomology as a contractible space, it can be an algebraic model of EG.

Let M be a left G-manifold.

An algebraic model for M is $\Omega(M) = \{C^\infty$ forms on $M\}$.

Thus, an algebraic model for $EG \times M$ is $W(\mathfrak{g}) \otimes \Omega(M)$.

An algeraic model for $M_G = (EG \times M)/G$ is $(W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}}$.

Def. If $\mathcal{A}$ is a G-dga, then $\mathcal{A}_{\text{hor}} = \{$horizontal elements$\} = \{\alpha \in \mathcal{A} | \iota_A\alpha = 0\ \forall A \in \mathfrak{g}\}$,

$\mathcal{A}_{\text{inv}} = \{$invariant elements$\} = \{\alpha \in \mathcal{A} | \mathcal{L}_A\alpha = 0\ \forall A \in \mathfrak{g}\}$ (Assuming G connected),

$\mathcal{A}_{\text{bas}} = \{$basic elements$\} = \{\alpha \in \mathcal{A} | \iota_A\alpha = 0, \mathcal{L}_A\alpha = 0\ \forall A \in \mathfrak{g}\}$.

We can extend d and ι_A to $W(\mathfrak{g}) \otimes \Omega(M)$ as antiderivations: $d(\alpha \otimes \beta) = (d\alpha) \otimes \beta + (-1)^{\deg\alpha}\alpha \otimes d\beta$, and $\mathcal{L}_A$ to $W(\mathfrak{g}) \otimes \Omega(M)$ as a derivation.

This makes $W(\mathfrak{g}) \otimes \Omega(M)$ into a G-dga.

Theorem (Equivariant de Rham theorem). For a connected Lie group G and a left G-manifold M,

$$H^*_G(M) \simeq H^*\{(W(\mathfrak{g}) \otimes_{\mathbb{R}} \Omega(M))_{\text{bas}}, d\},$$

where RHS gives a *Cartan model*.

Proposition: If $\mathcal{A}$ is a G-dga, then $d\mathcal{A}_{\text{bas}} \subset \mathcal{A}_{\text{bas}}$.

Prof: Suppose $\alpha \in \mathcal{A}_{\text{bas}}$ is basic. Then for $A \in \mathfrak{g}$, $\iota_A(d\alpha) = (\mathcal{A}_A - d\iota_A)\alpha = 0$ because α is basic.

$\mathcal{L}_A(d\alpha) = d(\mathcal{A}_\alpha) = 0$.

So $d\alpha$ is basic.

Example. Take $M = pt$. Then $H^*_G(pt) = H^*((EG \times pt)/G) = H^*(BG)$.

By the equivarinat de Rham theorem, $H^*(BG) = H^*\{W(\mathfrak{g})_{\text{bas}}, d\}$

$W(\mathfrak{g}) = \Lambda(\theta_1, ..., \theta_n) \otimes \mathbb{R}[u_1, ..., u_n] = \{a_0 + \sum_i a_i\theta_i + \sum_{i<j} a_{ij}\theta_i\theta_j + \cdots + a_{1\dots n}\theta_1 \cdots \theta_n | a_I \in \mathbb{R}[u_1, ..., u_n]\}$.

$\alpha \in W(\mathfrak{g})$ is horizontal iff $\iota_A\alpha = 0$ for all $A \in \mathfrak{g}$.

$\iota_A a_0 = 0$ because $\iota_A(\text{const.}) = 0$ and $\iota_A u_i = 0$. $\iota_{X_j}(\sum_i a_i\theta_i) = \sum_i a_i\delta_{ij} = a_j = 0$ if $\iota_A\alpha = 0$ if $\iota_A\alpha = 0$.

$\iota_{X_1}(\sum_{i<j} a_{ij}\theta_i\theta_j) = \iota_{X_1}(\sum a_{1j}\theta_1\theta_j + \sum_{2\le i<j} a_{ij}\theta_i\theta_j) = \sum_{1<j} a_{1j}\theta_j = 0$ if $\iota_{X_1}\alpha = 0$.

By induction, α is horizontal in $W(\mathfrak{g})$ if and only if $\alpha = a_0 \in \mathbb{R}[u_1, ..., u_n]$.

Thus $W(\mathfrak{g})_{\text{hor}} = \mathbb{R}[u_1, ..., u_n] = S(\mathfrak{g}^\vee)$, the symmetric algebra of the dual of $\mathfrak{g}$.

$W(\mathfrak{g})_{\text{bas}} = S(\mathfrak{g}^\vee)^G$,

By the equivariant de Rhan theorem, $H^*(BG) = H^*\{S(\mathfrak{g}^\vee)^G, d\}$.

Cartan model for a circle action:

Below we take the example of $G = S^1$, $\mathfrak{g} = \mathbb{R}$.

Let X be a nonzero element of $\mathfrak{g}$, θ its dual basis for $\mathfrak{g}^\vee$ in $\Lambda(g^\vee)$, u its dual basis in $S^(\mathfrak{g}^\vee)$.

Then, for $G = S^1$, $\mathfrak{g} = \mathbb{R}$, $W(\mathfrak{g}) = \Lambda(\theta) \otimes \mathbb{R}[u] = (\mathbb{R} \oplus \mathbb{R}\theta) \otimes \mathbb{R}[u] = \mathbb{R}[u] \oplus \theta\mathbb{R}[u] = \{a + \theta b | a, b \in \mathbb{R}[u]\}$.

Let M be an S^1-manifold, $W(\mathfrak{g}) \otimes_{\mathbb{R}} \Omega(M) = (\mathbb{R}[u] \oplus \theta\mathbb{R}[u]) \otimes_{\mathbb{R}} \Omega(M) = \Omega(M)[u] \oplus \theta\Omega(M)[u] = \{a + \theta b | a, b \in \Omega(M)[u]\}$.

$a + \theta b$ is horizontal iff $\iota_X(a + \theta b) = 0 \Leftrightarrow \iota_X a + b - \theta\iota_X b = 0 \Leftrightarrow b = -\iota_X a$, so $a + \theta b$ is horizontal iff

$$a + \theta b = a - \theta\iota_X a = (1 - \theta\iota_X)a$$

for $a \in \Omega(M)[u]$.

1.21 Lecture 21: The Cartan model for a circle action

First: a correction to the last lecture (which has been put back to the end of Lecture 19.)

Why is $W(\mathfrak{g})$ is a good model for EG?

$EG \to BG$ is the unique (up to G-homotopy) G-bundle s.t. for every principal G-bundle $P \to M$, there is a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \big\downarrow & & \big\downarrow \\ M & \longrightarrow & BG \end{array}$$

If EG were a manifold, then there would be a homomorphism $\Omega(EG) \to \Omega(P)$ for every principal G-bundle P, meaning that:

Every principal G-bundle $P \to M$ can be given a connection.

So there is a homomorphism (te Weil map) $W(\mathfrak{g}) \to \Omega(P)$, $\theta_k \mapsto \omega_k$, $u_k \mapsto \Omega_k$.

Moreoever, $W(\mathfrak{g})$ has the cohomology of a point. In this sense, $W(\mathfrak{g})$ is an algebraic model for EG.

Weil model for a G-manifold M is $W(\mathfrak{g}0 \otimes \Omega(M)$ $((W\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}}, d)$.

Horizontal elements for a circle action:

$G = S^1$, $\mathfrak{g} = i\mathbb{R}$ ($i = \sqrt{-1}$, we are putting G to be the circle $|z| = 1$, so $\mathfrak{g}$ is the line $z = 1$, which is $i\mathbb{R}$.) Let $X \neq 0$ in $\mathfrak{g}$. θ the dual basis for $\mathfrak{g}^\vee \subset \Lambda(\mathfrak{g}^\vee)$, u the dual basis for $\mathfrak{g}^\vee \subset S(\mathfrak{g}^\vee)$, then $W(\mathfrak{g}) = \Lambda(\theta) \otimes_{\mathbb{R}} \mathbb{R}[u] = (\mathbb{R} \oplus \mathbb{R}\theta) \otimes_{\mathbb{R}} \mathbb{R}[u] = \mathbb{R}[u] \oplus \mathbb{R}[u]\theta$.

Thus, $W(\mathfrak{g}) \otimes \Omega(M) = (\mathbb{R}[u] \otimes \mathbb{R}[u]\theta) \otimes_{\mathbb{R}} \Omega(M) = \Omega(M)[u] \oplus \Omega(M)[u]\theta$.

So an element of the Weil model $W(\mathfrak{g}) \otimes \Omega(M)$ is of the form $a + \theta b$, where $a, b \in \Omega(M)[u]$.

$a + \theta b$ is horizontal iff $\iota_X(a + \theta b) = 0 \Leftrightarrow b = -\iota_X a$ (we showed this in the last lecture).

So $(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{hor}} = \{a - \theta\iota_X a | a \in \Omega(M)[u]\}$.

Since G is connected, $a - \theta\iota_X a$ (see the theorems in Lcture 13 &14) is basic iff $a - \theta\iota_X a$ is S^1-invariant, i.e. iff $\mathcal{L}_X(a - \theta\iota_X a) = 0$.

$\mathcal{L}_X\theta = (d\iota_X + \iota_X d)\theta = d(1) + \iota_X u = 0$; $\mathcal{L}_X(\theta\iota_X a) = 0 + \theta\mathcal{L}_X\iota_X a = \theta\iota_X\mathcal{L}_X a$.

Then, $a - \theta\iota_X a$ is basic iff $\mathcal{L}_X a = 0$, i.e. iff a is S^1-invariant.

But $a \in \Omega(M)[u]$, so $a = a_0 + a_1 u + \cdots + \cdots a_k u^k$, where $a_i \in \Omega(M)$.

So $\mathcal{L}_X u = (d\iota_X + \iota_X d)u = 0$.

Thus, $\mathcal{L}_X a = 0$ iff $\mathcal{L}_X a_i = 0$ for all i, i.e. iff $a_i \in \Omega(M)^{S^1} = \{S^1\text{-invariant forms on } M\}$.

Finally, for $G = S^1$, and a G-manifold M, we have

$$(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} = \{a - \theta\iota_X a | a \in \Omega(M)^{S^1}[u]\}.$$

We have the *Weil–Cartan isomorphism*:

$$(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} \xrightarrow{\sim} \Omega(M)^{S^1}[u], \quad (1 - \theta\iota_X)a \leftrightarrow a.$$

The Cartan differential:

$$\begin{array}{ccc} (W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} & \xleftarrow{\lambda} & \Omega(M)^{S^1}[u] \\ \big\downarrow d & & \big\downarrow d_X \\ (W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} & \xrightarrow{\varphi} & \Omega(M)^{S^1}[u] \end{array}$$

The Cartan differential d_X is the linear map corresponding to the Weil differential d under the Weil–Cartan isomorphism.

Let $a \in \Omega(M)^{S^1}[u]$, then $d_X a = (\varphi \circ d \circ \lambda)(a) = (\varphi \circ d)(a - \theta\iota_X a) = \varphi(da - u\iota_X a + \theta d\iota_X a) = \varphi((da - u\iota_X a) - \theta\iota_X da) = \varphi((da - u\iota_X a) - \theta\iota_X(da - u\iota_X a)) = da - u\iota_X a$.

Thus we have shown that the Cartan differential is

$$d_X = d - u\iota_X.$$

Definition (Cartan model). The *Cartan model* is defined as $(\Omega(M)S^1[u], d_X)$.

Theorem (Equivariant de Rham theorem for $G = S^1$) We have $H^*_{S^1}(M) = H^*\{\Omega(M)^{S^1}[u], d_X\}$.

1.22 Lecture 22: Circle actions. Localization

Example: S^1 acting on S^2 by rotation about the z axis.

The volume form on S^2 is $\omega = xdy \wedge dz - ydx \wedge z + zdx \wedge dy$.

Choose X, the generator of S^1: $\mathfrak{g} = i\mathbb{R}$, let $X = -2\pi i$. Then the fundamental vector field $\underline{X}$ is defined by, at point (x, y, z), $\underline{X}_{(x,y,z)} = \frac{d}{dt}\Big|_{t=0} e^{-2\pi it} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t & 0 \\ \sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\pi y \frac{\partial}{\partial x} + 2\pi x \frac{\partial}{\partial y}$.

(Exercise: check $\mathcal{L}_{\underline{X}}\omega = 0$, so that ω is S^1-invariant.)

Definition (Equivariant differential form). A element of $\Omega(M)^{S^1}[u]$ is called an *equivariant differential form.*

If $\widetilde{\omega} \in \Omega(M)^{S^1}[u]$ has degree 2, then $\widetilde{\omega} = \omega_2 + fu$, where $\omega_2 \in \Omega^2(M)^{S^1}$ and $f \in \Omega^0(M)^{S^1}$. We say that $\widetilde{\omega}$ is an *equivariant extension* of ω_2.

$\widetilde{\omega}$ is *equivariantly closed* iff $d_X\widetilde{\omega} = 0$.

An equivariantly closed extension of the volume form:

Let $\widetilde{\omega} = \omega + fu$. $d_X\widetilde{\Omega} = d\widetilde{\omega} - u\iota_X\widetilde{\omega} = d\omega + (df)u - u\iota_X\omega - \underbrace{u\iota_X(fu)}_{=0} = d\omega + u(df - \iota_X\omega) = 0$ iff $d\omega = 0$ and $df = \iota_X\omega$. Since $d\omega = 0$ already, so the only condition for $d_X\widetilde{\omega} = 0$ is $df = \iota_X\omega$. We have $\iota_X\omega = \iota_{\underline{X}}\omega = 2\pi(x^2 + y^2 + z^2)dz - 2\pi z(xdx + ydy + zdz) = 2\pi dz = d(2\pi z)$. Thus $\widetilde{\omega} = \omega + 2\pi zu$ is equivariantly closed; it is the equivariant extension of the volume form.

Now, our goal is to finally obtain the coefficients a, b in Eq. (1.4). We need more theorems:

[In commutative algebra, localization means introducing denominators.]

Definition (Localization). Let N be an $\mathbb{R}[u]$-module. ($H^*_{S^1}(pt) = H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{R}[u]$.) Denote $\mathbb{N} = \{0, 1, 2, ...\}$ Define the *localization* of N as $N_u = \{\frac{x}{u^m} | x \in N, m \in \mathbb{N}\}/\sim$, where $\frac{x}{u^m} \sim \frac{y}{u^n} \Leftrightarrow$ if $\exists k \in \mathbb{N}$ s.t. $u^k(u^n x - u^m y) = 0$.

Example: $\mathbb{R}[u]_u = \{\frac{a_{-m}}{u^m} + \frac{a_{-m+1}}{u^{m-1}} + \cdots + \frac{a_{-1}}{u} + a_0 + a_1 u + \cdots + a_k u^k | a_i \in \mathbb{R}\} = \{\text{Laurent polynimials in } u\} = \mathbb{R}[u][u^{-1}] = \mathbb{R}[u, u^{-1}]$.

N_u becomes an $\mathbb{R}[u]$-module.

There is a $\mathbb{R}[u]$-module homomorphism $i\colon N \to N_u$, $x \mapsto \frac{x}{1}$. Its kernel is $\ker i = \{x \in N | \frac{x}{1} \sim \frac{0}{1}\} = \{x \in N | \exists k \in \mathbb{N} \text{ s.t. } u^k x = 0\} = \{u\text{-torsion elements in } N\}$.

Hence, there is an exact sequence $0 \to \{u\text{-torsion elements in } N\} \to N \xrightarrow{i} N_u \to 0$.

Definition (Torsion module). N is u-torsion if every element $x \in N$ is a u-torsion.

[The use of localization: if localize N to N_u and we get zero, $N_u = 0$, then we know that N is u-torsion. This is the theorem below:]

Proposition. N is u-torsion iff $N_u = 0$.

(Proof: "$\Leftarrow$" If $N_u = 0$, then the exact sequence gives that N is u-torsion. "$\Rightarrow$": Suppose N is u-torsion, let $\frac{x}{u^m} \in N_u$, there exists $k \in \mathbb{N}$ s.t. $u^k x = 0$, so $\frac{x}{u^m} \sim \frac{u^k x}{u^k u^m} = \frac{0}{u^k u^m} = 0$, so $N_u = 0$.)

Theorem 1. If S^1 acts freely on M, then $H^*_{S^1}(M)$ is u-torsion.

Proof:

$$\begin{array}{ccccc} ES^1 & \longleftarrow & ES^1 \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ BS^1 & \longleftarrow & M_{S^1} & \longrightarrow & M/S^1 \end{array}$$

Since S^1 acts freely on M, by the Cartan mixing diagram above, $M_{S^1} \to M/S^1$ is a fiber bundle with fiber ES^1.

By homotopy exact sequence, M_{S^1} is weakly homotopic to M/S^1. By a theorem of algebraic topology (see e..g. Hatcher) that says if two spaces are weakly homotopic, then they have the same cohomology, then we have $H^*(M_{S^1}) \simeq H^*(M/S^1)$. As $H^*_{S^1}(M) = H^*(M_{S^1})$, and $H^*(M/S^1)$ is the cohomology of some finite dimensional manifold, this means that $H^*(M/S^1)$ is a finite dimensional vector space over $\mathbb{R}$.

On the other hand, the equivariant cohomology $H^*_{S^1}(M)$ is a $\mathbb{R}[u]$-module ($M \to pt$ induces $H^*(pt) \to H^*_G(M)$, where $G = S^1$ so $H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{R}[u]$). We further assume M is compact, so $\mathbb{R}[u]$-module is finitely generated $\mathbb{R}[u]$-module. (Next lecture we will look at non-compact case.)

Since $\mathbb{R}[u]$ is a PID, and finitely generated modules over PID has the structure theorem which says that it has the form $\mathbb{R}[u]^r \oplus \text{torsion}$. If it contains nonzero $\mathbb{R}[u]^r$ then this would be infinite dimensional. Hence, $H^*_{S^1}(M)$ is torsion.

Next time we will further show that $H^*_{S^1}(M)$ is a u-torsion.

1.23 Lecture 23: Properties of localization

If $k > \dim(M/G)$, then $u^k \cdot H_G^*(M) = 0$ (since its degree is larger than the dimension of $\dim(M/G)$.)

If M/G is not compact, then $H_G^*(M)$ would be infinitely dimensional, but $u^k \cdot H_G^*(M) = 0$ still holds for a free action. Therefore we have the following result

Theorem. If S^1 acts freely on M, then $H_{S^1}^*(M)$ is u-torsion.

To generalize this result to non-free actions, we need further results from commutative algebra/homological algebra. Main three results we need are

1. Localization preserves exactness;
2. Localization commutes with quotient
3. Localization commutes with cohomology.

Theorem 1. If $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of $\mathbb{R}[u]$ modules at B, then $A_u \xrightarrow{f_u} B_u \xrightarrow{g_u} C_u$ is exact at B_u.

(Recall that $f_u(\frac{a}{u^m}) = \frac{f(a)}{u^m}$, so $1_u = 1$, $(g \circ f)_u = g_u \circ f_u$, so $\mathrm{im} f_u \subset \ker g_u$; Suppose $g_u(\frac{b}{u^m}) \sim \frac{0}{1}$, i.e. $\frac{g(b)}{u^m} \sim \frac{0}{1}$, so there exists $k \in \mathbb{N}$ s.t. $u^k \cdot g(b) = 0$. But g is a morphism of $\mathbb{R}[u]$-modules, so $u^k g(b) = g(u^k b) = 0$, so $u^k b \in \ker g = \mathrm{im} f$, so $u^k b = f(a)$ for some $a \in A$. so $b = \frac{f(a)}{u^k}$, and $\frac{b}{u^m} = \frac{f(a)}{u^{k+m}} = f_u(\frac{a}{u^{k+m}})$, this proves that $\ker g_u \subset \mathrm{im} f_u$. Therefore $A_u \to B_u \to C_u$ is exact at B_u. The result can them be extended to longer exact sequences.)

Theorem 2. Localization commutes with quotient: If A is an $\mathbb{R}[u]$-submodule of B, then $\left(\frac{B}{A}\right) u \simeq \frac{B_u}{A_u}$.

($0 \to A \to B \to B/A \to 0$ is exact. Since localization preserves exactnes, $0 \to A_u \to B_u \xrightarrow{g} (B/A)_u \to 0$ is exact. This implies that (by one of the isomorphism theorems of algebra) $\frac{B_u}{\ker g} \simeq \mathrm{im} g$ and so $\frac{B_u}{A_u} \simeq \left(\frac{B}{A}\right)_u$.)

Theorem 3. Localization commutes with cohomology: If $\mathcal{C}$ is a differential complex, $C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots$ s.t. $d^2 = 0$, then $\mathcal{C}_u\colon C_u^0 \xrightarrow{d_u} C_u^1 \xrightarrow{d_u} C_u^2 \to \cdots$ is again a differential complex (because $d_u^2 = (d^2)_u = 0$), and $H^*(\mathcal{C}_u, d_u) \simeq H^*(\mathcal{C}, d)_u$.

($H^k(\mathcal{C}_u, d_u) = \frac{\ker(d_u\colon C_u^k \to C_u^{k+1})}{\mathrm{im} d_u\colon C_u^{k-1} \to C_u^k}$.

We have $0 \to \ker d \to C^k \xrightarrow{d} C^{k+1}$ is exact. Since localization preserves exactness, so $0 \to (\ker d)_u \to C_u^k \xrightarrow{d_u} C_u^{k+1}$ is exact, so we have $(\ker d)_u \cong \ker(d_u)$.

On the other hand, $C^{k+1} \xrightarrow{d} C^k \to C^k/\mathrm{im} d \to 0$ is exact. So when localize, we get that $C_u^{k+1} \xrightarrow{d_u} C_u^k \xrightarrow{\pi} (C^k/\mathrm{im} d)_u \to 0$ is exact, and we have $(C^k/\mathrm{im} d)_u \simeq C_u^k/(\mathrm{im} d)_u$ by theorem 1; and by the isomorphism theorem we have $\mathrm{im}\pi = \frac{C_u^k}{\ker\pi} = \frac{C_u^k}{\mathrm{im}(d_u)}$, so $\mathrm{im}(d_u) \cong (\mathrm{im} d)_u$.

Therefore $H^k(\mathcal{C}_u, d_u) = \frac{\ker d_u}{\mathrm{im} d_u} = \frac{(\ker d)_u}{(\mathrm{im} d)_u} \simeq \left(\frac{\ker}{\mathrm{im} d}\right)_u = H^k(\mathcal{C}, d)_u$.)

Next lecture we will study locally free action:

Definition (Locally free action). An action of a topological group G on a topological space X is *locally free* if $\mathrm{Stab}(x)$ is discrete for any $x \in X$.

Example: S^1 acts on $\mathbb{C}^2$ by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^n z_2)$, where $n \in \mathbb{Z}^+$.

$\mathrm{Stab}(0,0) = S^1$, $\mathrm{Stab}(0, z_2) = \{n\text{-th roots of } 1\}$ if $z_2 \neq 0$, and $\mathrm{Stab}(z_1, z_2) = 1$ if $z_1 \neq 0$.

The action of S^1 on $\mathbb{C}^2 - \{(0,0)\}$ is locally free.

1.24 Lecture 24: Cohomology of a locally free action

Recall the definition of locally free action in the last lecture.

Proposition. If a Lie group G acts smoothly and locally-freely on a manifold M, then for all $X \neq 0 \in \mathfrak{g}$, the fundamental vector field $\underline{X}$ is nowhere vanishing on M.

(Example: S^1 acts locally freely on $S^2 = \{p, q\}$, where p is the north pole and q the south pole.)

(Proof: Suppose $\underline{X}_p = 0$ for some $p \in M$. Then $c(t) = p$ is an integral curve of $\underline{X}$ through p: $c'(t) = 0 = \underline{X}_p = \underline{X}_{c(t)}$. But $\varphi(t) = e^{-tX} \cdot p$ is also an integral curve for $\underline{X}$ through p. By the uniqueness of the integral curve through p, $e^{-tX} \cdot p = p$ for all $t \in \mathbb{R}$. So $\mathrm{Stab}(p)$ contains a curve e^{-tX}, so is not discrete. This contradicts the locally-free condition.)

Proposition. If a compact abelian group acts locally freely on a manifold M, and $X \neq 0 \in \mathfrak{g}$, then there exists a G-invariant 1-form φ on M s.t. $\varphi(\underline{X}) = 1$.

(we will use the following fact: A manifold with a compact Lie group action has an invariant Riemannian metric $\langle\ ,\ \rangle$.)

(Proof: Define for all $p \in M$, $\varphi_p\colon T_pM \to \mathbb{R}$ by $\varphi_p(Z_p) = \frac{\langle \underline{X}_p, Z_p \rangle}{||\underline{X}_0||^2}$, where $\langle\ ,\ \rangle$ is a G-invariant Riemannian metric. Check that $\varphi_p(\underline{X}_p) = 1$. Let $g \in G$. $(l_g^*\varphi)_p(Z_p) = \varphi_{gp}(l_{g*}Z_p) = \frac{\langle \underline{X}_{gp}, l_{g*}Z_p \rangle}{\langle \underline{X}_{gp}, \underline{X}_{gp} \rangle} = \frac{\langle l_{g*}\underline{X}_p, l_{g*}\underline{Z}_p \rangle}{\langle \langle l_{g*}\underline{X}_p, \langle l_{g*}\underline{X}_p \rangle} = \frac{\langle \underline{X}_p, \underline{Z}_p \rangle}{\langle \langle \underline{X}_p, \langle \underline{X}_p \rangle} = \varphi_p(Z_p)$, where we used $l_{g*}(\underline{X}_p) = \underline{(\mathrm{Ad} g) X}_{gp} = \underline{X}_{gp}$ because G is abelian, and the fact that the metric in G-invariant. So we showed that $l_g^*\varphi = \varphi$. Hence φ is G-invariant.)

The Cartan model for an S^1-action on M is $(\Omega(M)^{S^1}[u], d_X)$, where $d_X\alpha = d\alpha - u\iota_X\alpha$.

(Check that 1. $d_X\colon \Omega(M)^{S^1}[u] \to \Omega(M)^{S^1}[u]$ is an antiderivation of degree +1. 2. $d\colon N \to N$ is an antiderivation of $\mathbb{R}[u]$-algebras, then $d_u\colon N_u \to N_u$ is an antidervation of $\mathbb{R}[u,u^{-1}]$-algebras.)

Theorem. If S^1 acts locally freely on a manifold M, then $H^*_{S^1}(M)$ is u-torion.

Proof. It suffices to show $H^*_{S^1}(M)_u = 0$. By the equivariant de Rham theorem, $H^*_{S^1}(M) \cong H^*\{\Omega(M)^{S^1}[u], d_X\}$, where $X \neq 0 \in \mathrm{Lie}(S^1)$. So $H^*_{S^1}(M)_u \cong H^*\{\Omega(M)^{S^1}[u,u^{-1}], (d_X)_u\}$ (because localization commutes with the cohomology).

We have a $\varphi \in \Omega^1(M)^{S^1}$ s.t. $\varphi(\underline{X}) = 1$. To show $H^*\{\Omega(M)^{S^1}[u,u^{-1}], (d_X)_u\}$ is zero, it suffices to find an $\alpha \in \Omega(M^{S^1}[u,u^{-1}]$ s.t. $(d_X)_u\alpha = 1$.

Then $d_X\left(\frac{\alpha}{u^m}\right) = (d_X)_u\left(\frac{\alpha}{u^m}\right) = \frac{d_X\alpha}{u^m}$, If $d_X\alpha = 1$ and z is any d_X-cocycle, then $z = (d_X\alpha)z = d_X(\alpha z)$ (because $d_X z = 0$), meaning that any cocycle z is a coboundary.

To let's aim to find this α. $d_X\varphi = d\varphi - u\iota_X\varphi = d\varphi - u$ (because $\iota_X\varphi = \iota_{\underline{X}}\varphi = \varphi(\underline{X}) = 1$).

So $\frac{d_X\varphi}{u} = \frac{d\varphi}{u} - 1$, giving $-(1-d\varphi/u)^{-1}\frac{d_X\varphi}{u} = 1$, or $-\left(1 + \frac{d\varphi}{u} + \cdots + \frac{(d\varphi)^{n-1}}{u^{n-1}}\right)\frac{d_X\varphi}{u} = 1$, where the expansion stops because $(d\varphi)^n$ alwady has degree $2n > n$, where $n := \dim M \neq 0$. So $(d\varphi)^n = 0$ on M.

$(d_X)_u = \frac{d_X(d\varphi)}{u} = \frac{d(d\varphi) - u\iota_X d\varphi}{u} = d\iota_X\varphi = d(1) = 0$, where we abbreviated $(d_X)_u = d_X$, and we used $\iota_X d = \mathcal{L}_X - d\iota_X$ and $\mathcal{L}_X\varphi = 0$ because φ is an invariant form.

Thus, $1 + \frac{d\varphi}{u} + \cdots + \frac{(d\varphi)^{n-1}}{u^{n-1}}$ is a d_X-cocycle.

$d_X\varphi u = d_X\left(\frac{\varphi}{u}\right)$.

So $-d_X\left((1 + \frac{d\varphi}{u} + \cdots + \frac{(d\varphi)^{n-1}}{u^{n-1}})\frac{\varphi}{u}\right) = 1$. Let $\alpha = (1 + \frac{d\varphi}{u} + \cdots + \frac{(d\varphi)^{n-1}}{u^{n-1}})\frac{\varphi}{u}$ we have $d_X\alpha = 1$, then $H^*\{\Omega(M)^{S^1}[u,u^{-1}], (d_X)_u\} = H^*_{S^1}(M)_u = 0$.

This proves the theorem.

1.25 Lecture 25: General facts about $H^*_G(M)$

[Part of the proof of the last theorem has been moved to the last subsection.]

Theorem (Borel localization theorem for a circle action). If S^1 acts on a manifold M with compact fixed point set F. then

$$H^*_{S^1}(M)_u \xrightarrow{\cong} H^*_{S^1}(F)_u$$

is an isomorphism of algebras over $\mathbb{R}[u,u^{-1}]$.

(Eventually of course we want to get rid of the localization constraint. We will get there later.)

Above fixed point sets, we have:

Proposition: The fixed point set F of a continuous group action on a Hausdorff topological space X is closed in X.

(Proof: Let p be a limit point of F, i.e. there is a sequence $p_n \in F$ s.t. $\lim p_n = p$. Then, $\forall g \in G$, $g \circ p_n = p_n$. Because the action is continuous, then $g \circ p_n \to g \circ p$. So $p_n \to p = g \cdot p$, therefore $p \in F$. Thus F is closed.

Theorem. If a compact Lie group acts smoothly on a manifold M, then its fixed point set F is a regular submanifold of M. (A regular submanifold is the same as embedded submanifold; and is very different from the immersed submanifold.)

Proof. Since G is compact, we can put a G-invariant Riemannian metric on M. For $x \in M$, consider the exponential map $\mathrm{Exp}_x\colon V \subset T_xM \xrightarrow{\sim} U \subset M$. By choosing V sufficiently small, $\mathrm{Exp}_x\colon V \to U$ is a diffeomorphism. We can then use $(\mathrm{Exp}_x)^{-1}$ as a coordinate map on U.

T_xM has a given inner product. This makes T_xM into a Riemannian manifold. G acts on T_xM by $g \cdot v = l_{g*}v \in T_xM$ since $g \cdot x = x$.

We can choose $V \subset T_xM$ so that V is G-invariant. For example, if $V = B(0,\varepsilon)$ and $v \in V$, then $||l_{g*}v||^2 = \langle l_{g*}v, l_{g*}v\rangle = \langle v, v\rangle = ||v||^2 < \varepsilon^2$. So $l_{g*}v \in B(0,\varepsilon)$. Thus , any open ball centered at 0 in T_xM is G-invariant. Choose a sufficiently small open ball to be V. Then, from diffirential geometry, there is a commutative diagram $\begin{array}{ccc} V & \xrightarrow{l_{g*}} & V \\ \downarrow{\scriptstyle \mathrm{Exp}_x} & & \downarrow{\scriptstyle \mathrm{Exp}_x} \\ U & \xrightarrow{l_g} & U \end{array}$.

Because $l_g\colon U \to U$ is an isometry, $\langle l_{g*}w, l_{g*}v\rangle = \langle w, v\rangle$. Let F be the fixed point set of G on M. Then $F \cap U$ is the fixed point set of G on U.

$\mathrm{Exp}_x^{-1}(F \cap U)$ is the fixed point set of G on V. The subset of V fixed l_{g*} is $\{v \in V | l_{g*}v = v\} = V \cap E_g$, where E_g is the eigenspace of l_{g*} with eigenvalue 1

So $\mathrm{Exp}_x^{-1}(F \cap U) = V \cap (\cap_{g \in G} E_g) = V \cap (\text{linear subspace}) \simeq F \cap U$,

Hence F is a regular submanifold of M.

This completes the proof.

1.26 Lecture 26: Equivariant tubular neighborhood and equivariant Mayer–Vietoris

Definition (Equivariant vector bundle). A vector bundle $\pi \colon E \to M$ is G-equivariant if (i) both E and M are left G-spaces, and $\pi \colon E \to M$ is G-equivariant. (i.e. if $x \in M$ goes to gx, then the fiber at x goes to the fiber at gx.) (ii) G acts on each fiber linearly, i.e. $l_g \colon E_x \to E_{gx}$ is a linear transformation for all $g \in G$, $x \in M$. (Here we introduced the notation $E_x := \pi^{-1}(x)$.)

Proposition. If $\pi \colon E \to M$ is a G-equivariant vector bundle with fiber V, then $\pi_G \colon E_G \to M_G$ is a vector bundle with fiber V.

Def. (Tubular neighborhood) A *tubular neighborhood* of a submanifold $S \subset M$ of a manifold M is an open set U containing S s.t. U has the structure of a vector bundle over S with the inclusion $i \colon S \hookrightarrow U$ being the zero section.

Definition (Equivariant tubular neighborhood). A *G-equivariant tubular neighborhood* of a G-invariant submanifold $S \subset M$ in a G-manifold M is a G-invariant open set U containing S s.t. U has the structure of a G-equivariant vector bundle over S with the inclusion $i \colon S \hookrightarrow U$ being the zero section.

Theorem. (Tubular neighborhood theorem) If $S \subset M$ is a compact submanifold, then S has a tubular neighborhood U s.t. $U \to S$ is isomorphic to the normal bundle $N_{S/M}$ of S in M.

(Proof is given in Spivak's book.)

Theorem (Equivariant tubular neighborhood theorem). If $S \subset M$ is a compact G-invariant submanifold of a G-manifold M, then S has a G-equivariant tubular neighborhood U s.t. $U \to S$ is isomorphic to $N_{S/M}$.

Theorem (Equivariant Mayer–Vietoris sequence). Let M be a G-manifold, U, V two G-invariant open subsets such that $M = U \cup V$. Then there is an exact sequence

$$\cdots \to H_G^{k-1}(U \cap V) \to H_G^k(M) \to H_G^k(U) \oplus H_G^k(V) \xrightarrow{r} H_G^k(U \cap V) \to H_G^{k+1}(M) \to \cdots \tag{1.7}$$

where the map r is $r \colon (\alpha, \beta) \mapsto (i^U_{U\cap V})^*\alpha - (i^V_{U\cap V})^*\beta$.

Lemma. If U, V are G-invariant open sets that cover a G-manifold M, then U_G, V_G are open sets that cover M_G.

(Proof. Since $i \colon U \to M$ is G-equivariant and injective, then $i_G \colon U_G \to M_G$ is injective (by the property of equivariant map). By definition, $U_G = (EG \times U)/G$. Since $EG \times U$ is open on $EG \times M$, so U_G is open in M_G. Similarly, V_G is open in M_G. Then, we claim that $M_G = U_G \cup V_G$: let $[e, x] := [(e, x)] \in M_G$, where $(e, x) \in EG \times M$. So $x \in M = U \cup V$. If $x \in U$, then $[e, x] \in U_G$, and if $x \in V$, then $[e, x] \in V_G$. Therefore $[e, x] \in U_G \cup V_G$. This shows that $M_G = U_G \cup V_G$.)

With this lemma, we can appy the ordinary Mayer–Vietoris sequence provided we have the following lemma:

Lemma. $U_G \cap V_G = (U \cap V)_G$.

(Proof: $U \cap V \hookrightarrow U$ is G-equivariant and injective, so $(U \cap V)_G \to U_G$ is injective. Similarly, $(U \cap V)_G \to V_G$ is injective. So $(U \cap V)_G \subset U_G \cap V_G$. Let $[e, x] \in U_G \cap V_G$. Then $[e, x] \in U_G$, so $(eg, g^{-1}x) \in EG \times U$ for some $g \in G$. Since U is G-invariant, so $x \in U$. Similarly $x \in V$. So $x \in U \cap V$. So $[e, x] \in (U \cap V)_G$, so $U_G \cap V_G \subset (U \cap V)_G$.)

Now we apply the usual Mayer–Vietoris sequence to the open cover $\{U_G, V_G\}$ of M_G, we get

$$\cdots \to H^{k-1}(U_G \cap V_G) \to H^k(M_G) \to H^k(U_G) \oplus H^k(V_G) \to H^k(U_G \cap V_G) \to H^{k+1}(M_G) \to \cdots$$

which is Eq. (1.7) using the fact that $H_G^*(U \cap V) = H^*(U_G \cap V_G)$, $H^k(M_G) = H_G^k(M)$ and so on.

Theorem. If S^1 acts on M with no fixed points, then the action is locally free.

Proof: Since there are no fixed points for any $x \in M$, $\mathrm{Stab}(x)$ is a proper subgroup of S^1. Note that $\mathrm{Stab}(x)$ is a closed subgroup (because if g is a limit of $\mathrm{Stab}(x)$ then $\exists g_n \in \mathrm{Stab}(x)$ s.t. $g_n \to g$, because the action is continuous, $g_n \cdot x \to g \cdot x$, which is $x \to x$ as $g_n \in \mathrm{Stab}(x)$. By the uniqueness of limit, $g \cdot x = x$, so $g \in \mathrm{Stab}(x)$.). As a closed subgroup of the Lie group S^1, $\mathrm{Stab}(x)$ is a regular submanifold. If $\dim\mathrm{Stab}(x) = 1$, so $\mathrm{Stab}(x)$ is open in S^1. But $\mathrm{Stab}(x)$ is closed, and S^1 is connected, so $\mathrm{Stab}(x)$ is either $\emptyset$ or S^1 itself, both impossible. Then we must have $\dim\mathrm{Stab}(x) = 0$, then $\mathrm{Stab}(x)$ is discrete, hence the S^1 action is locally free.

1.27 Lecture 27: Borel localization for circle action

The Borel localization theorem (given at the beginning of Lecture 25): Suppose S^1 acts on a manifold M with complact fixed point set F. Then the restriction map is a ring isomorphism: $H^*_{S^1}(M)_u \xrightarrow{\sim} H^*_{S^1}(F)_u$.

Proof. Denote $G = S^1$ in this proof. Since F is a closed subset of a compact manifold M, it is compact. From last lecture, F is a compact submanifold. Since F is G-invariant, it has an equivariant tubular neighborhood N_F, which is isomorphic to the normal bundle $N_{F/M}$. $M - F$ is a G-invariant open subset, s.t. $\{N_F, M - F\}$ is an open cover of M by G-invariant open sets.

By the equivariant Mayer–Vietoris sequence, we have a long exact sequence

$$\cdots \to H_G^{k-1}(N_F \cap (M-F)) \to H_G^k(M) \to H_G^k(N_F) \oplus H_G^k(M-F) \to H_G^k(N_F \cap (M-F)) \to \cdots$$

[Since localization preserves exactness, we are tempted to write down

$$\cdots \to H_G^{k-1}(N_F \cap (M-F))_u \to H_G^k(M)_u \to H_G^k(N_F)_u \oplus H_G^k(M-F)_u \to H_G^k(N_F \cap (M-F))_u \to \cdots,$$

but this actually is an exact sequence of $\mathbb{R}$-modules, not of $\mathbb{R}[u]$ modules. This shows we cannot localize each term with respect to u.]

The equivariant Mayer–Vietoris sequence can be written in the form

$$\begin{tikzcd} H^*_{S^1}(M) = \bigoplus_k H^k_{S^1}(M) \arrow[rr, "i^*"] & & H^*_{S^1}(N_F) \oplus H^*_{S^1}(M-F) \arrow[dl, "j^*"] \\ & H^*(N_F \cap (M-F)) \arrow[ul, "\delta"] & \end{tikzcd}$$

where all three terms are $\mathbb{R}[u]$-modules, and i^*, j^*, δ are $\mathbb{R}[u]$-homomorphisms; i^* and j^* are of degree 0, δ is of degree 1.

This triangule is exact in the sense that the kernel of any map is the image of the prceding map.

[This is an example of exact couple – see Lecture 21 of Bott–Tu.]

Since each term is an $\mathbb{R}[u]$-module, we can localize with respect to u: as localization preserves exactness we have the exact triangle

$$\begin{tikzcd} H^*_{S^1}(M)_u \arrow[rr, "i^*"] & & H^*_{S^1}(N_F)_u \oplus H^*_{S^1}(M-F) \arrow[dl, "j^*"] \\ & H^*(N_F \cap (M-F))_u \arrow[ul, "\delta"] & \end{tikzcd}$$

As S^1 acts on $M-F$ with no fixed points, so the action is locally free, so we have $H^*_G(M-F)_u = 0$ and $H^*_G(N_F \cap (M-F)) = 0$. Therefore the restriction map $i^* \colon H^k_{S^1}(M)_u \to H^k_{S^1}(N_F)_u$ is an isomorphism. But $H^k_{S^1}(N_F) = H^k((N_F)_{S^1})$; Next we'd like to show that $H^k((N_F)_{S^1}) \simeq H^*(F_G)$.

Since N_F is an equivariant tubular neighborhood of F, $N_F \to F$ is G-equivariant vector bundle, Hence $(N_F)_G \to F_G$ is a vector bundle, in which $F_G \hookrightarrow (N_F)_G$ is the zeroth section.

For any vector bundle $E \to M$, there is a deformation retraction from E to the zero section. So $H^*((N_F)_G) \simeq H^*(F_G)$,

Therefore we have an isomorphism

$$H^k_G(M)_u \to H^k_G(N_F)_u = H^k_{S^1}(F)_u.$$

We have proved the Borel localization theorem.

Remark: the Borel localization theorem holds for a noncompact manifold M as long as the fixed point set F is compact.

In general, the inclusion $i \colon F \hookrightarrow M$ induces the restriction i^* that fits into an exact sequence

$$0 \to \ker i^* \to H^*_{S^1}(M) \xrightarrow{i^*} H^*_{S^1}(F) \to \operatorname{coker} i^* \to 0,$$

where $\operatorname{coker} i^* = H^*_{S^1}(F)/\operatorname{im} i^*$.

Then we have the exact sequence for the localization

$$0 \to \ker i^*_u \to H^*_{S^1}(M)_u \xrightarrow{i^*} H^*_{S^1}(F)_u \to \operatorname{coker} i^*_u \to 0.$$

The Borel localization theorem tells us that

Corollary. Suppose S^1 acts on a manifold M with compact fixed point set F. Then

(i) $\ker i^* \colon H^*_{S^1}(M) \to H^*_{S^1}(F)$ and $\operatorname{coker} i^*$ are u-torsion.

(ii) if in addition, $H^*_{S^1}(M)$ is a free finitely generated $\mathbb{R}[u]$-module, then $\ker i^* = 0$.

(Proof of (ii): $H^*_{S^1}(M)$ is a free finitely generated $\mathbb{R}[u]$-module over $\mathbb{R}[u]$ which is a PID, so it is free; but it is also a u-torsion by (i), so we have $\ker i^* = 0$.)

Remark: (ii) can be replaced by "if in addition, $H^*_{S^1}(M)$ is torsion-free, then $\ker i^* = 0$."

1.28 Lecture 28: Borel localization and ring structure

[First, a correction to the proof of the Borel localization theorem. This part has been incorporated in the proof in the last lecture.]

The ring structure of $H^*_{S^1}(S^2)$, where S^1 aicts on S^2 by rotation about the z axis.

From the spectral sequence of $\begin{array}{ccc} S^2 & \longrightarrow & (S^2)_{S^1} \\ & & \downarrow \\ & & BS^1 = \mathbb{C}P^\infty \end{array}$, which degenerates at E_2, we have $H^*_{S^1}(S^2) = E_\infty = E_2 \cong H^*(\mathbb{C}P^\infty) \otimes H^*(S^2) = \mathbb{R}[u] \otimes (\mathbb{R} \oplus \mathbb{R}\omega) = \mathbb{R}[u] \oplus \mathbb{R}[u]\widetilde{\omega}$, where ω is the volumne form on S^2.

We have $d\omega = 0$ but $d_X\omega = d\omega - u\iota_X\omega \neq 0$, so ω is not an equivariantly closed form in $H^*_{S^1}(S^2)$. But we found that an equivariant closed form extension of ω is $\widetilde{\omega} = \omega + (2\pi z)u$. This is why we had to use $\widetilde{\omega}$ as the basis.

Let $a = \frac{\widetilde{\omega}}{2\pi} = \frac{\omega}{2\pi} + zu$. So we have $H^*_{S^1}(S^2) = \mathbb{R}[u] \oplus \mathbb{R}[u]a = \mathbb{R}[u,a]/(a^2 - u(c_1u + c_2a))$.

$H^*_{S^1}(S^2)$ is generated as a ring over $\mathbb{R}$ by u and a; We need to determine $a^2 = c_1u^2 + c_2ua$.

Below we use the Borel localization theorem.

The fixed point set $F = \{p, q\}$ (north and south poles), we have an exact sequence

$$0 \to \ker i^* \to H^*_{S^1}(S^2) \xrightarrow{i^*} H^*_{S^1}(F),$$

By borel localization, $(\ker i^*)_u = 0$, so $\ker i^*$ is torsion.

Since $H^*_{S^1}(S^2)$ is torsion-free, so $\ker i^*$ (as a submodule) is torsion-free.

So $\ker i^* = 0$, and $H^*_{S^1}(S^2) \hookrightarrow H^*_{S^1}(F)$.

But $H^*_{S^1}(F) = H^*_{S^1}(\{p,q\}) = H^*(p_{S^1} \amalg q_{S^1}) = H^*(p_{S^1}) \oplus H^*(q_{S^1}) = H^*(BS^1) \oplus H^*(BS^1) = \mathbb{R}[u_p] \oplus \mathbb{R}[u_q]$.

We have $i^*u = (i_p^*u, i_q^*u) = (u_p, u_q)$, $i^*a = (i_p^*a, i_q^*a) = (u_p, -u_q)$ (using the expression of a, and the fact that the restriction of the 2-form ω, $i^*\omega = 0$ on a single point).

Hence $i^*(a^2) = (u_p^2, u_q^2)$, $u^*(u^2) = (u_p^2, u_q^2)$, so $i^*(a^2 - u^2) = (0,0)$; Since $i^*\colon H^*_{S^1}(S^2) \to H^*_{S^1}(F)$ is injective, we have $a^2 - u^2 = 0$ in $H^*_{S^1}(S^2)$.

1.29 Lecture 29: Ring structure continued; Local data at a fixed point

$H^*_{S^1}(S^2)$ is generated as a polynomial ring over $\mathbb{R}$ y u, a, with relation $a^2 - u^2 = 0$ and maybe other relations.

So there is a ring homomorphism

$$0 \to \ker\alpha \to \frac{\mathbb{R}[u,a]}{(a^2 - u^2)} \xrightarrow{\alpha} H^*_{S^1}(S^2) \to 0. \tag{1.8}$$

Now, $\mathbb{R}[u]$ is a PID, and as an $\mathbb{R}[u]$-module, $\frac{\mathbb{R}[u,a]}{(a^2-u^2)} = \mathbb{R}[u] \oplus \mathbb{R}[u]a$ is a free module.

Theorem. A submodule S of a free module M (not necessarily finitely generated) over a PID is free, and $\mathrm{rk}S \leq \mathrm{rk}M$. (see e.g. Rotman)

Therefore, $\ker\alpha$ is free and of $\mathrm{rk} \leq 2$.

But $H^*_{S^1}(S^2)$ is also a free $\mathbb{R}[u]$-module of rank 2. As Eq. (1.8) is an exact sequence of free modules, the middle module is the direct sum of the left and right. This shows that $\ker\alpha$ has rank 0, so $\ker\alpha = 0$.

Therefore

$$\frac{\mathbb{R}[u,a]}{(a^2 - u^2)} \xrightarrow{\alpha} H^*_{S^1}(S^2)$$

is a ring isomorphism.

Suppose a Lie group G ats on a manifold M smoothly on the left. Then for any $g \in G$, $l_g\colon M \to M$ is a diffeomorphism. So there is an isomorphism $l_{g*}\colon T_xM \to T_{gx}M$. If x is a fixed point of the action, then $l_{g*}\colon T_xM \to T_xM$. This gives a map $\rho\colon G \to \mathrm{GL}(T_xM) = \{$non-singular linear automorphisms of $T_xM \to T_xM\}$, that sends $g \mapsto l_{g*}$.

Moreoever, $\rho(gh) = (l_g \circ l_h)_* = l_{g*} \circ l_{h*} = \rho(g) \circ \rho(h)$.

Thus $\rho\colon G \to \mathrm{GL}(T_xM)$ is a group homomorphism.

Def. A *representation* of G is a group homomoprhism $\rho\colon G \to \mathrm{GL}(V)$ for some vector space V.

Def. If $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ are representations, then $\rho_1 \oplus \rho_2\colon G \to \mathrm{GL}(V_1 \oplus V_2)$ is defined by $(\rho_1 \oplus \rho_2)(g)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \rho_1(g)v_1 \\ \rho_2(g)v_2 \end{pmatrix} = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Def. W is an *invariant subspace* of V if $\rho(g)(W) \subset W$.

Def. If 0 and V are the only invariant subspaces, then $\rho\colon G \to \mathrm{GL}(V)$ is *irreducible*. Otherwise, ρ is *reducible*. ρ is *completely reducible* if ρ is the direct sum of irreducible representations.

(Q: is every reducible representation completely reducible?)

Example: $\rho\colon \mathbb{R} \to \mathrm{GL}(\mathbb{R}^2)$ by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is reducible, because x-axis is an invariant subspace. But it is not completely reducible, as the matrix is not diagonalizable.

Theorem. Every finite dimensional representation of a complact Lie group is completely reducible.

Def. Two representations $\rho_V\colon G \to \mathrm{GL}(V)$ and $\rho_W\colon G \to \mathrm{GL}(W)$ are *equivalent* if there exists an isomorphism $f\colon V \to W$ s.t. $\forall g \in G$,

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "f", "\simeq"'] & V \arrow[d, "f", "\simeq"'] \\ W \arrow[r, "\rho_W(g)"] & W \end{tikzcd}$$

is commutative.

Theorem. The nonequivalent irreducible representations of S^1 are (1) the trivial rep $1\colon S^1 \to \{1\} \subset \mathrm{GL}(\mathbb{R}) = \mathbb{R}^\times$ of dimension 1; (2) rotation $L^m\colon S^1 \to \mathrm{GL}(\mathbb{R}^2)$, $L^m(e^it) = \begin{pmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{pmatrix}$, $m \in \mathbb{Z}^+$. (In general, L^m is equivalent to L^{-m}.)

Isolated fixed points of a circle action

At a fixed point x of a circle action on M, We have $\rho\colon S^1 \to \mathrm{GL}(T_xM)$, so ρ can be written as a direct sum of irreducible representations of S^1:

$\rho = L^{m_1} \oplus L^{m_2} \oplus \cdots \oplus L^{m_k} \oplus 1 \oplus \cdots \oplus 1.$

Theorem. If x is an isolated fixed point of a circle action, then ρ does not contain any trivial representation as summands.

Proof. Since S^1 is compact, there exists an S^1-invariant Riemannian metric on M. Then $l_g\colon M \to M$ is an isometry, $\langle l_{g*}v, l_{g*}w\rangle = \langle v, w\rangle$ for all $g \in G$.

At a fixed point $x \in M$, choose V to be a small open ball about $0 \in T_xM$,

Then there exists a commutative diagram

$$\begin{tikzcd} V \arrow[r, "l_{g*}"] \arrow[d, "\mathrm{Exp}_x"'] & V \arrow[d, "\mathrm{Exp}_x"] \\ U \arrow[r, "l_{g*}"] & U \end{tikzcd}$$

where Exp_x is a diffeomorphism.

So $l_g(\mathrm{Exp}_x v) = \mathrm{Exp}_x(l_{g*}v)$, if $l_{g*}v = v$ for all $g \in G$, then $\mathrm{Exp}_x v$ is a fixed point of G.

Under the diffeomorphism, Fixed points in $V \xrightarrow{\sim} F \cap U$,

But Fixed points in $V = V \cap$ (linear subspace of T_xM)

If $F \cap U$ is isolated, then $\cap_{g\in G}$(eigenspaces of l_{g*} with eigenvalue 1) $= \{0\}$, therefore ρ cannot contain a trivial representation of dimension 1.

1.30 Lecture 30: Localization formula for a circle action

For a circle action, $\int_M \phi = \sum_{p\in F} \iota_p$ for an equivariantly closed form ϕ.

Manifolds with boundary

Def. If M is a manifold with boundary ∂M, and $p \in \partial M$, then locally at p, there exists a neighborhood U s.t. U is homeomorphic to open subset of $\mathbb{H}^n = \{(x^1, ..., x^n) \in \mathbb{R}^n | x^n \geq 0\}$.

$T_pM = \{$derivatives on germs of C^∞ functions at $p\} = \mathbb{R}\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^n}|_p\}$. $T_p^*M = \mathbb{R}\{dx^1|_p, ..., dx^n|_p\}$.

Everything that we have done so far generalizes to a manifold with boundary.

Integration of an equivariantform:

Def. If $\omega \in \Omega(M)^{S^1}[u]$ is an equivariant form of degree k, then

$\omega = \omega_k + \omega_{k-2}u + \omega_{k-4}u^2 + \cdots$, and we define

$$\int_M \omega = \int_M \omega_k + \left(\int_M \omega_{k-2}\right) u + \left(\int_M \omega_{k-4}\right) u^2 + \cdots$$

If k and $n = \dim M$ have different parity: then $\int_M \omega = 0$.

If k and $n = \dim M$ have the same parity, say $k = n + 2m$, $\int_M \omega = \begin{cases} \left(\int_M \omega_n\right) u^m & \text{for } k \geq n, \\ 0, & \text{for } k < n. \end{cases}$

Theorem (Stoke's theorem for equivariant forms). Suppose S^1 acts smoothly on a compact orientable manifold M with boundary ∂M. (S^1 will act on the ∂M.) If $\omega \in \Omega(M)^{S^1}[u]$ of degree k, then

$$\int_M d_X\omega = \int_{\partial M} \omega.$$

(k is independent of $n = \dim M$.)

Proof. Suppose $k+1$ and n have different parity, then both sides are zero for degree reasons. Now assume $k+1 = n+2m$. Then $\int_M d_X\omega = \int_M d\omega - u\iota_X\omega = \left(\int_M d\omega_{n-1}\right)u^m - u^{\#}\int_M \iota_X\omega_{n+1} = \left(\int_M d\omega_{n-1}\right)u^m = \left(\int_{\partial M}\omega_{n-1}\right)u^m = \int_{\partial M}\omega$. (we have used that ω_{n+1} is automatically zero on a manifold M with dimension n.)

Rationale for a localization theorem

Suppose S^1 acts on M with only isolated fixed points. Let $F = \{\text{fixed points}\}$.

Then T_pM has no trivial 1-dimensional irreducible representations. So $T_pM = L^{m_1} \oplus \cdots \oplus L^{m_n}$, where L is the standard 2-dimensional representation of S^1. Hence, $\dim M = 2n$.

We can put an S^1-invariant Riemannian metric on M. Then S^1 acts by isometries on M.

Around each point p, let $B(p,\varepsilon)$ be an open ball of radius ε. S^1 acts on $M - \cup_{p\in F}B(p,\varepsilon)$ without fixed points, and therefore the action is locally free. Let $X = 2\pi i \in \mathrm{Lie}(S^1)$. For a locally free action, we found an S^1-invariant 1-form $\theta \in \Omega^1(M)^{S^1}$, s.t. $\theta(\underline{X}) = 1$. And $\alpha \in \Omega(M)^{S^1}[u,u^{-1}]$ s.t. $d_X\alpha = 1$. In fact, $\alpha = -\frac{\theta}{u}\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right)$, α has degree -1.

With this α, we can define a cochain homotopy $K\colon \Omega(N)^{S^1}[u,u^{-1}] \to \Omega(N)^{S^1}[u,u^{-1}]$ by $K\omega = \alpha\omega$. Check: $Kd_X + d_XK = 1$.

Now replace M by $M = \cup_{p\in F}B(p,\varepsilon)$, if ϕ is equivariantly closed on M, then $\phi = (Kd_X + d_XK)\phi = d_XK\phi$ on $M - \cup B(p,\varepsilon)$, so $\int_{M-\cup B(p,\varepsilon)}\phi = \int_{M-\cup B(p,\varepsilon)} d_XK\phi = -\int_{\partial(M-\cup B(p,\varepsilon))}K\phi = \sum_{p\in F}\int_{S_p^{2n-1}(\varepsilon)}K\phi$. (To be continued in the next lecture.)

1.31 Lecture 31: the ABBV localization formula for a cicle action

Then, we have $\int_M \phi = \lim\limits_{\varepsilon\to 0}\int_{M-\cup B(p,\varepsilon), p\in F}\phi = \sum\limits_{p\in F} c_p$.

At an isolated fixed point p, $T_pM = L^{m_1}\oplus\cdots\oplus L^{m_n}$, the numbers $m_1,...,m_n$ are the *exponents* of the fixed point, defined up to sign. But with the requirement that the orientations on the two sides agree.

Theorem (Atiyah–Bott 1984, Berline–Vergne 1982). If S^1 acts on a compact oriented manifold M of dimension $2n$, with only isolated fixed point set F, and $\phi = \phi_{2n} + \phi_{2n-2}u + \cdots + fu^n \in \Omega(M)^{S^1}[u]$, $\deg\phi = 2n$, ϕ is equivariantly closed, then

$$\int_M \phi_{2n} = \int_M \phi = \sum_{p\ inF}\frac{f(p)}{m_1\cdots m_n(p)}.$$

Conditions for ϕ to be equivarinatly closed:

$d_X\phi = d\phi - u\iota_X\phi = 0 \Leftrightarrow d_X\phi = d\phi_{2n} + (d\phi_{2n-2})u + \cdots - \iota_X\phi_{2n})u - (\iota_X\phi_{2n-2})u^2 - \cdots = 0 \Leftrightarrow d\phi_{2n} = 0, d\phi_{2n-2} = \iota_X\phi_{2n}$, $d\phi_{2n-4} = \iota_X\phi_{2n-2}, ..., df = \iota_X\phi_2$.

Example. Surface area of unit sphere S^2.

Let S^1 act on S^2 by rotation about z axis. Let $X = 2\pi i \in \mathrm{Lie}(S^1)$. The volume form on S^2 is $\omega = xdy\wedge dz - ydx\wedge dz + zdx\wedge dy \in \Omega^2(S^2)^{S^1}$.

$\mathrm{Area}(S^2) = \int_M \omega$.

We need an equivariantly closed form $\widetilde{\omega} = \omega + fu$. $d_X\widetilde{\omega} = 0 \Leftrightarrow d\omega = 0$ and $\iota_X\omega = df$.

We found before that $f = 2\pi z$.

We orient S^2 using ω in the sense that if v_1, v_2 is a positive basis for T_pM, then $\omega_p(v_1,v_2) > 0$. So the orientation of S^2 at $P = (0,0,1)$ is given by $dx\wedge dy$. Hence the orientation for T_PS^2 is $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Hence $T_PS^2 = L$ so $m(P) = 1$.

At the south pole $Q = (0,0,-1)$, $\omega_Q = -dx\wedge dy$, so $T_Q(S^2)$ is oriented by $(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$.

Hence $m(Q) = -1$.

By the ABBV formula, $\mathrm{Area}(S^2) = \int_{S^2}\omega = \int_{S^2}\widetilde{\omega} = \frac{f(P)}{m(P)} + \frac{f(Q)}{m(Q)} = \frac{2\pi\cdot 1}{1} + \frac{2\pi\cdot(-1)}{-1} = 4\pi$.

Blow-ups (a way to avoid taking limits)

Definition (Blow-up). The *blow-up* of a manifold M at a point $p\in M$ is $(\widetilde{M}, \sigma\colon \widetilde{M}\to M)$ where $\widetilde{M}$ is manifold with boundary and $\sigma\colon \widetilde{M} - \partial\widetilde{M} \to M - \{p\}$ is a diffeomorphism, and $\sigma^{-1}(p) = \partial\widetilde{M}$ is in one-to-one correspondence with the unit sphere in T_pM.

Let B be open subset of $\mathbb{R}^2$ with $0\in B$. Define $\widetilde{B} = \{(x,v)\in B\times S^{2n-1} | x \text{ is in the ray of } v\}$. Then define $\sigma\colon \widetilde{B}\to B$, $\sigma(x,v) = x$. Then $\sigma^{-1}(0) = \{(0,v)|v\in S^1\} = S^1$. If $x\neq 0$, then $\sigma^{-1}(x) = (x, \frac{x}{|x|})$, so σ is a diffeomorphism on $\widetilde{B} - \sigma^{-1}(0)$ and maps S^1 to 0.

1.32 Lecture 32: Proof of the localization formula, continued

Real blow-up:

If $\sigma\colon \widetilde{M}\to M$ is the real blow-up of M at $p\in M$. Locally relative to a charge $(U,x^1,...,x^n)$ at p, $\sigma\colon \widetilde{U}\to U$ is given by $\widetilde{U}=\{(x,v)\in U\times S^{n-1}|x=tv$ for some $t\in\mathbb{R}^{\geq 0}, v\in S^{n-1}$ is a unit vector$\}$. If $x\neq 0$, then it determines a unique inverse image $(x,\frac{x}{||x||})$; If $x=0$, then $\sigma^{-1}(0)=\{(0,v)|v\in S^{n-1}\}$.

If $f\colon M\to M$ has p as a fixed point, then f induces a map $\widetilde{f}\colon\widetilde{M}\to\widetilde{M}$: on $\widetilde{M}-\sigma^{-1}(p)$, σ is a diffeomorphism, so $\widetilde{f}=\sigma^{-1}\circ f\circ\sigma$; on $\sigma^{-1}(p)=S^{n-1}=\{$all tangent directions at $p\}$, by continuity, $\widetilde{f}(v)=\frac{f_*(v)}{||f_*(v)||}$.

Suppose S^1 acts on M with isolated fixed points. Put an S^1-invariant metric on M, $\dim M=2n$. Then S^1 acts on M by isometries.

At a fixed point p, T_pM is a representation of S^1, so $T_pM=L^{m_1}\oplus\cdots\oplus L^{m_n}\neq 0$. Where L is the standard representation of S^1 on $\mathbb{R}^2$. (e^{it} acts on (x,y) through the usual rotation matrix $\mathrm{rot}(t)$.)

The action of S^1 on M induces an action on $\widetilde{M}$ that takes S_p^{2n-1} to $S_p^{2n-1}\subset T_pM$. If $w=(u_1,v_1,u_2,v_2,...,u_n,v_n)\in T_pM$, then $e^{it\cdot w}=\mathrm{diag}(\mathrm{rot}(m_1t),\mathrm{rot}(m_2t),...,\mathrm{rot}(m_nt))w$.

Let $X=-2\pi i\in\mathrm{Lie}(S^1)$, then $\underline{X}_w=\frac{d}{dt}\Big|_{t=0}e^{2\pi it}\cdot w=\frac{d}{dt}\Big|_{t=0}\mathrm{diag}(\mathrm{rot}(2\pi m_1t),...,\mathrm{rot}(2\pi m_nt))w=\mathrm{diag}(2\pi m_1J,...,2\pi m_nJ)w$, where we defined $J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. So $\underline{X}_w=2\pi\sum_{j=1}^n m_j\left(-v_j\frac{\partial}{\partial u_j}+u_v\frac{\partial}{\partial v_j}\right)$ on T_pM.

A 1-form θ on T_pM s.t. $\theta(\underline{X})=1$ is $\theta=\left(\frac{1}{2\pi}\sum\frac{1}{m_j}(-v_jdu_j+u_jdv_j)\right)\frac{1}{\sum_{j=1}^n(u_j^2+v_j^2)}$.

On S_p^{2n-1}, $\underline{X}=2\pi\sum m_j\left(-v_j\frac{\partial}{\partial u_j}+u_v\frac{\partial}{\partial v_j}\right)$, $\theta=\frac{1}{2\pi}\sum\frac{1}{m_j}(-v_jdu_j+u_jdv_j)$.

Volume form on a sphere:

$S^{n-1}\in\mathbb{R}^n$ is the boundary of the n-ball D^n. D^n can be the same orientation as $\mathbb{R}^n$. $\mathrm{vol}_{D^n}=dx^1\wedge\cdots\wedge dx^n$, we give S^{n-1} the boundary orientation of D^n. The radial vector is $\vec{r}=\sum x^i\frac{\partial}{\partial x^i}$. Its volume form is $\mathrm{vol}_{S^1}=\iota_{\vec{r}}\mathrm{vol}_{D^n}=\iota_{\sum x^i\partial_{x^i}}(dx^1\wedge\cdots\wedge dx^n)=\sum_{i=1}^n(-1)^{i-1}dx^1\wedge\cdot\wedge\hat{dx^i}\wedge\cdot\wedge x^n$.

Surface area of S^{2n-1}: $\mathrm{vol}(S^{2n-1})=\int_{S^{2n-1}}\mathrm{vol}_{S^{2n-1}}=\frac{2\pi^n}{(n-1)!}$.

Proof of the localization formula

Suppose S^1 acts on M with isolated fixed point set F. Let $\sigma\colon\widetilde{M}\to M$ be the blow-up at all the fixed points. Choose $X\neq 0\in\mathrm{Lie}(S^1)$. Let $\phi\in\Omega(M)^{S^1}[u]$.

Lemma: $d_X\sigma^*=\sigma^*d_X$ on $\Omega(M)^{S^1}[u]$.

(Proof: $d_X\sigma^*\phi=d\sigma^*\phi-u\iota_X\sigma^*\phi=\sigma^*d\phi-u\iota_X\sigma^*\phi$; $(\iota_X\sigma^*\phi)(...)=\sigma^*\phi(\underline{X}_{\widetilde{M}},...)=\phi(\sigma_*\underline{X}_{\widetilde{M}},\sigma_*...)=\phi(\underline{X}_M,\sigma_*...)=(\iota_X\phi)(\sigma_*,...)=(\sigma^*\iota_X\phi)(...)$. Therefore, $d_X\sigma^*\phi=\sigma^*d\phi-u\sigma^*\iota_X\phi=\sigma^*d_X\phi$.)

Assume ϕ equivariantly closed. We have found $K\colon\Omega(\widetilde{M})^{S^1}[u,u^{-1}]\to\Omega(\widetilde{M})^{S^1}[u,u^{-1}]$ s.t. $K\omega=\alpha\omega$, where we defined $\alpha=-\frac{\theta}{u}\left(1+\frac{d\theta}{u}+\cdots+\left(\frac{d\theta}{u}\right)^{n-1}\right)$, and $d_XK+Kd_X=1$. So $d_XK\phi=\phi$.

1.33 Lecture 33: Completing the proof of the localization formula

$\int_M\phi=\int_{M-F}\phi$ (because F has measure 0) $=\int_{\widetilde{M}-\partial M}\sigma^*\phi$ (because $\sigma\colon\widetilde{M}-\partial\widetilde{M}\to M-F$ is a diffeomorphism) $=\int_{\widetilde{M}}\sigma^*\phi$ (because $\partial\widetilde{M}$ has measure zero) $=\int_{\widetilde{M}}d_XK\sigma^*\phi$ (since $\sigma^*\phi$ is equivarinatly closed) $=\int_{\partial\widetilde{M}}K\sigma^*\phi$ (Stoke's theorem) $=\int_{\cup_{p\in F}-S_p^{2n-1}}K\sigma^*\phi=\sum_{p\in F}\int_{S_p^{2n-1}}\frac{\theta}{u}\left(1+\frac{d\theta}{u}+\cdots+\left(\frac{d\theta}{u}\right)^{n-1}\right)\sigma^*\phi$.

Assume $\deg\phi=2n$, then $\phi=\phi_{2n}+\phi_{2n-2}u+\cdots+fu^n\in\Omega(M)^{S^1}[u]$, $\int_M\phi=\int_M\phi_{2n}$.

In the integral $\int_{S_p^{2n-1}}K\sigma^*\phi=\int_{S_p^{2n-1}}i^*(\alpha\sigma^*\phi)=\int_{S_p^{2n-1}}(i^*\alpha)i^*\sigma^*\phi$, $\sigma^*\phi$ is restricted to S_p^{2n-1}, so
$$\begin{array}{ccc}S_p^{2n-1}&\xrightarrow{i}&\widetilde{M}\\ \sigma\downarrow&&\downarrow\sigma\\ \{p\}&\xrightarrow{i}&M\end{array},$$
so $i^*\sigma^*\phi=\sigma^*i^*\phi=\sigma^*(f(p)u^n)=f(p)u^n$,

So $\int_M\phi=\sum_{p\in F}\int_{S_p^{2n-1}}\frac{\theta}{u}\left(1+\frac{d\theta}{u}+\cdots+\left(\frac{d\theta}{u}\right)^{n-1}\right)f(p)u^n=\sum_{p\in F}\int_{S_p^{2n-1}}\theta(d\theta)^{n-1}f(p)=\sum_{p\in F}f(p)\int_{S_p^{2n-1}}\theta(d\theta)^{n-1}$.

Recall that $\theta=\frac{1}{2\pi}\sum_{j=1}^n\frac{1}{m_j}(-g_jdu_j+u_jdv_j)$ on S_p^{2n-1}, $d\theta=\frac{1}{\pi}\sum\frac{1}{m_j}du_j\wedge dv_j$,

$(d\theta)^{n-1}=\frac{(n-1)!}{\pi^{n-1}}\sum_{j=1}^n\frac{du_1dv_1...\hat{du_j}\hat{dv_j}...du_ndv_n}{m_1...\hat{m_j}...m_n}$,

$\theta(d\theta)^{n-2}=\frac{1}{2\pi^n}\sum\frac{1}{m_1...m_n}(v_jdu_1dv_1...du_j\hat{dv_j}...du_ndv_n+u_jdu_1dv_1...\hat{du_j}dv_j...du_ndv_n=\frac{(n-1)!}{2\pi^nm_1...,m_n(p)}\mathrm{vol}_{S_p^{2n-1}}$.

Hence $\int_{S_p^{2n-1}}\theta(d\theta)^{n-1}=\frac{(n-1)!}{2\pi^n}\frac{1}{m_1...m_n(p)}\frac{2\pi^n}{(n-1)!}=\frac{1}{m_1...m_n(p)}$.

Finally, $\int_M\phi=\sum\limits_{p\in F}\frac{f(p)}{m_1...m_n(p)}$, which is the localization fomula.

Note that $\int_M \phi_{2n} = \int_M \phi$. So this gives a way to calculate the ordinary dfferential form.

Later part of the lecture introduces "the Cartan model in general" which we put to next lecture.

1.34 Lecture 34: The Cartan model in general

Let G be a connected Lie group acting on a manifold M on the left.

Choose a basis $X_1, ..., X_n$ for $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\theta_1, ..., \theta_n$ be the dual basis for $\mathfrak{g}^\vee$ in $\Lambda(g^\vee)$. Let $u_1, ..., u_n$ be a dual basis for $\mathfrak{g}^\vee$ in $S(\mathfrak{g}^\vee)$, where $\deg\theta_i = 1$ and $\deg u_i = 2$.

For $X \in \mathfrak{g}$, $\iota_X\theta_i = \theta_i(X)$, $\iota_X u_i = 0$, $d\theta_k = -\frac{1}{2}\sum c_{ij}^k\theta_i \wedge \theta_j + u_k$, $du_k = \sum c_{ij}^k u_i\theta_j$.

$\mathcal{L}_X = d\iota_X + \iota_X d$.

Introduce the shorthand: $\iota_i = \iota_{X_i}$, $\mathcal{L}_i = \mathcal{L}_{X_i}$, $a_I = a_{i_1 \ldots i_m}$, and $\theta_I = \theta_{i_1} \wedge \cdots \theta_{i_m} = \theta_{i_1,,,i_m}$.

The *Weil algebra* is $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^\vee) \otimes_{\mathbb{R}} S(\mathfrak{g}^\vee) = \Lambda(\theta_1, ..., \theta_n) \otimes \mathbb{R}[u_1, ..., u_n]$.

An algebraic model for $EG \times M$ is $W(\mathfrak{g}) \otimes \Omega(M)$, with $\iota_X\theta_i = \theta_i(X)$, $\iota_X u_i = 0$.

An element of $W(\mathfrak{g}) \otimes \Omega(M) = \Lambda(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) \otimes \Omega(M)$ can be written as

$$a = a_0 + \sum_i \theta_i a_i + \sum_{i<j} \theta_i \wedge \theta_j a_{ij} + \cdots + \theta_1 \wedge \cdots \wedge \theta_n a_{1\ldots n},$$

where $a_I \in S(\mathfrak{g}^\vee) \otimes \Omega(M)$,

An element $a \in W(\mathfrak{g}) \otimes \Omega(M)$ is *horizontal* if $\iota_X a = 0$ for all $X \in \mathfrak{g}$, and is *invariant* if $\mathcal{L}_X a = 0$ for all $X \in \mathfrak{g}$, and is *basic* if it is both horizontal and invariant.

Because ι_X and $\mathcal{L}_X$ are $\mathbb{R}$-linear in X, a is horizontal $\Leftrightarrow$ $\iota_{X_i} a = 0$ for $i = 1, ..., n$, and a is invariant $\Leftrightarrow$ $\mathcal{L}_{X_i} a = 0$ for $i = 1, ..., n$.

ι_X, d extend to $W(\mathfrak{g} \otimes \Omega(M)$ as antiderivations, $\mathcal{L}_X$ extend to $W(\mathfrak{g} \otimes \Omega(M)$ as a derivataion.

Theorem: there is an algebra isomorphism:

$$(W(\mathfrak{g}) \otimes \Omega(m))_{\mathrm{hor}} \cong S(\mathfrak{g}^\vee) \otimes \Omega(M),$$

where $a = a_0 + \sum a_I\theta_I \mapsto a_0$.

(I.e. the horizontal elements are precisely those who do not have θ's.)

(Proof: $a_0 = \sum u_{i_1} \cdots u_{i_m}\omega_I$, $\iota_X a_0 = \sum u_{i_1} \cdots u_{i_m}\iota_X\omega_I = 0$, as $\iota_X\omega = 0$ for $\omega \in \Omega(M)$. For simplicity we show the case of $n = 2$: $a = a_0 + \theta_1 a_1 + \theta_2 a_2 + \theta_1 \wedge \theta_2 a_{12}$, $0 = \iota_{X_1} a = \iota_1 a_0 + a_i - \theta_1\iota_1 a_1 - \theta_2\iota_1 a_2 + \theta_2 a_{12} + \theta_1\theta_2\iota_1 a_{12}$, $0 = \iota_{X_2} a = \iota_2 a_0 - \theta_1\iota_2 a_1 + a_2 - \theta_2\iota_2 a_2 - \theta_1 a_{12} + \theta_1\theta_2\iota_2 a_{12}$. a is horizontal $\Leftrightarrow$ $a_1 = -\iota_1 a_0$ and $a_2 = -\iota_2 a_0$ and $a_{12} = \iota_1 a_2 = -\iota_2 a_1$, $\Leftrightarrow$ $a_0 = -\iota_1 a_0$ and $a_2 = -\iota_2 a_0$ and $a_{12} = \iota_2\iota_1 a_0$ $\Leftrightarrow$ $a = a_0 - \theta_1\iota_1 a_0 - \theta_2\iota_2 a_0 + \theta_1\theta_2\iota_2\iota_1 a_0 = (1 - \theta_1\iota_1)(1 - \theta_2\iota_2)a_0$. (In the case $n > 2$, there will just be more factors.) ϕ has an inverse $a_0 \mapsto (1 - \theta_1\iota_1)(1 - \theta_2\iota_2)a_0$, so ϕ is an algebra homomorphism, and therefore an algebra isomorphism.)

The above theorem implies that

$$\underbrace{(W(\mathfrak{g}) \otimes \Omega(m))_{\mathrm{bas}}}_{\text{Weil Model}} \cong \underbrace{(S(\mathfrak{g}^\vee) \otimes \Omega(M))^G}_{\text{Cartan Model}}.$$

The Weil model has a differential d_W. The isomorphism above induces a differential (the "Cartan differential") for the Cartan model.

1.35 Lecture 35: Applications of equivariant cohomology

The equivairant de Rham theorem is for any compact Lie group, but the localization theorems are for torus action.

(There's a localization theorem for a compact non-abelian group, by Lisa Jeffrey, Frances Kirwan. For noncompact non-abelian group, much is kunnown.)

Theorem (Localization theorem for a torus action). Suppose a torus T of dimensional l acts smoothly on a compact closed manifold M with fixed point set F (not necessarily isolated or 0-dimensional). If ϕ is equivariantly closed on M of any degree, then

$$\int_M \phi = \int_F \frac{i^*\phi}{e^T(\nu)},$$

where $i\colon F \hookrightarrow M$ is the inclusion, ν is the normal bundle of F in M, $e^T(-)$ is the equivariant Euler class in $H_T^{2\dim F}(F) = \mathbb{R}[u_1, ..., u_n]$.

A priori, the RHS is a rational function in $u_1, ..., u_l$. But because the LHS is a polynomial in $u_1, ..., u_l$, the rational function is actually a polynomial.

Application 1. Integral of invariant forms:

(see the example in previous lectures on the calculation of $\int_{S^2} \mathrm{vol}_{S^2}$.)

Application 2. Computation of topological invariants

Suppose G is a compact Lie group, and T is a maximal torus in G. E.g. $G = U(n)$, then $T = U(1) \times \cdots \times U_1 = S^1 \times \cdots \times S^1$. Then G/T = complete flag manifold of $\mathbb{C}^n = \{0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_n = \mathbb{C}^n,\ \text{where } \Lambda_i \simeq \mathbb{C}^i\}$.

$U(n)/U(1) \times \cdots U(1)$ has Chern numbers, T acts on G/T by $t \cdot (gT) = (tg)T$. gT is a fixed point of the action $\Leftrightarrow$ $tgT = gT$ forall $t \in T$, $\Leftrightarrow g^{-1}tgT = T$, $\Leftrightarrow g^{-1}tg \in T$, $\Leftrightarrow g^{-1}Tg = T$, $\Leftrightarrow g \in N_G(T)$, $\Leftrightarrow gT = N_G(T)/T := W_G(T)$, the *Weyl group* of T in G.

For Lie group G, the Weyl group $W_G(T)$ is a finite reflection group.

Chern number of G/T is $\int_{G/T} c_1(-)^{i_1} \cdots c_r(-)^{i_r}$, where the Chern classes are defined for the tangent bundle. The localization formula can be used to calculate this integral. We get $\int_{G/T} c_1(-)^{i_1} \cdots c_r(-)^{i_r} = \sum\limits_{w \in W_G(T)} \cdots$.

This can be generalized to a closed subgroup H of G of maximal rank, e.g. $\mathbb{C}P^n = \frac{U(n)}{U(1) \times U(n-1)}$, $G(k, \mathbb{C}^n) = \frac{U(n)}{U(k) \times U(n-k)}$.

(Rank of H is the dimension of the maximal torus of H. The rank of $U(n)$ is n.)

For example,

$$\int_{G(k,\mathbb{C}^n)} c_1(S)^{m_1} \cdots c_k(S)^{m_k} = \sum_{I=(i_1,...,i_k)} \frac{\prod_{r=1}^k \sigma_r(u_{i_1}, ..., u_{i_k})^{m_r}}{\prod_{i \in I} \prod_{j \in J} (u_i - u_j)},$$

where the Weyl group of $G(k, \mathbb{C}^n)$ is permutations, indexed by $I = (i_1, ..., i_k)$; and J = the complement of I in $(1, ..., n)$, ad σ_r is the r-th elementary symmetric function.

The RHS in fact is an integer as the LHS suggests.

(For more results, see L. Tu, Computing characteristic number using fixed points, `https://arxiv.org/abs/math/0102013`.)

Application 3. Identities. E.g. $\mathbb{C}P^2 = G(1, \mathbb{C}^3)$. $1 = \int_{\mathbb{C}P^2} (-h)^2 = \int_{\mathbb{C}P^2} c_1(S)^2 = \sum_{i \neq j = 1}^3 \frac{u_i^2}{\prod_{i<j}(u_i - u_j)}$.

Application 4. Pappus's theorem:

Use three parallel planes to intersept the sphere. If the two vertical distances are equal, then the surface areas are equal.

Let $\omega = \mathrm{vol}_{S^2}$. Calculate Let $\widetilde{\omega} = \omega + fu$ be the equivariant extension. We have
$\int_{z=a}^{z=b} \omega = \int_{z=a}^{z=b} \widetilde{\omega}$ = localization formula $= 2\pi(b-a)$.

Application 5. Symplectice geometry:

M is *symplectic* if it has a closed nondegenerate 2-form ω on it, and G acts symplectically on M if $l_g^* \omega = \omega$ for any $g \in G$. If G is connected, this is equivalent to the condition that $\mathcal{L}_X \omega = 0$. Then $d\iota_X \omega = (\mathcal{L}_X - \iota_X d)\omega = 0$, Hence $\iota_X \omega$ is closed.

If $\iota_X \omega$ is exact, say $\iota_X \omega = df$, then the action is said to be *Hamiltonian*.

Then $\widetilde{\omega} = \omega + fu$ is equivariantly closed. This Hamiltonian action is the Hamiltonian in classical mechanics. So we have the correspondence Classical mechanics $\Leftrightarrow$ Symplectic geometry $\Leftrightarrow$ equivariant cohomology.

6. Application in physics: $\int_M e^{itf} \tau$ = localization theorem $= \sum \cdots$

1.36 Lecture 36: The equivariant de Rham theorem

Recall that:

The Cartan model for a connected Lie group: $\big((S(\mathfrak{g}^\vee) \otimes \Omega(M))^G, D\big)$.

Let $X_1, ..., X_n$ be a basis for $\mathfrak{g} = \mathrm{Lieb}(G)$, $u_1, ..., u_n$ be the dual basis for $\mathfrak{g}^\vee$, then $S(\mathfrak{g}^\vee) = \mathbb{R}[u_1, ..., u_n]$. The Cartan differential D is given by $D\alpha = d\alpha - \sum u_i \iota_{X_i} \alpha$, where we define d s.t. $du_i = 0$.

Equivariant de Rham theorem: there is an algebra isomorphism $H_G^*(M; \mathbb{R}) \xrightarrow{\sim} H^*\{(S(\mathfrak{g}^\vee) \otimes \Omega(M))^G, D\}$.

(Cartan proves this theorem for a free action in 1950.)

G acts on $S(\mathfrak{g}^\vee)$ by the coadjoint representation, and G acts on $\Omega(M)$ by pulling back.

Corollary of the equivariant de Rham theorem. Take $M = pt$.

Then $H_G^*(pt) = H^*(BG; \mathbb{R}) \simeq H^*\{(S(\mathfrak{g}^\vee) \otimes)_{\mathbb{R}} \Omega(pt))^G, D\} = H^*\{S(\mathfrak{g}^\vee)^G, D\}$ by equivariant de Rham theorem; we have $D = 0$ on $S(\mathfrak{g}^\vee)^G$, so we have $H_G^*(pt) = H^*(BG; \mathbb{R}) \simeq S(\mathfrak{g}^\vee)^G \simeq S(\mathfrak{t}^\vee)^W$, where $\mathfrak{t}$ is the Lie algebra of the maximal torus T in G, and $W = N_G(T)/T$ is the Weykl group of T. (the step $S(\mathfrak{g}^\vee)^G \simeq S(\mathfrak{t}^\vee)^W$ is an easy theorem which can be found in L. Tu's 2010 paper).

To summarize, we have

$$H^*(BG) \simeq S(\mathfrak{g}^\vee)^G \simeq S(\mathfrak{t}^\vee)^W.$$

Example: Let $G = U(n)$, $T = U(1) \times \cdots \times U(1) = \{\mathrm{diag}(t_1, ..., t_n) | t_i \in U(1) = S^1\}$, then $\mathfrak{t} = \{\mathrm{diag}(*, ..., *) | * \in i\mathbb{R}\}$. Then $W = N_G(T)/T = \frac{\{\text{group generated by } T \text{ and } E_{ij}\}}{T} = S_n$ (Where we defined E_{ij}, the $n \times n$ matrix for transposition (ij), and S_n is the symmetric group on n letters.)

So

$$H^*(BU(n)) = S(\mathfrak{g}^\vee)^W = \mathbb{R}[u_1, ..., u_n]^{S_n},$$

the symmetric polynomials in $u_1, ..., u_n$.

Cartan's theorem on the cohomology of base of a principal bundle:

Theorem. If $P \to N$ is a principal G-bundle, then $H^*(N) = H^*\{(W(\mathfrak{g}) \otimes \Omega(P))_{\mathrm{bas}}\}$.

Assuming this theorem, one can prove equivariant de Rham theorem:

First, assume free actions:

Fact: If G is a compact Lie group acting freely on the right on a manifold M, then M/G is a manifold, and $M \to M/G$ is a principal G bundle. (For proof, see e.g. Lee's book on Manifolds.)

By Cartan's theorem above, $H^*(M/G) = H^*\{(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}}\} = H^*\{(S(\mathfrak{g}^\vee) \otimes \Omega(M))^G, D\}$, where the last one used the Weil–Cartan isomorphism. On the oter hand, when G acts freely, M_G and M/G are weakly homotopy equivalent, hene $H^*_G(M) = H^*(M_G) = H^*(M/G)$ (theorem in algebraic topology), so we have

$$H^*_G(M) = H^*\{(S(\mathfrak{g}^\vee) \otimes \Omega(M))^G, D\},$$

Proving the equivarianat de Rham theorem for a free G-action.

Equivariant de Rham theorem for an arbitrary action: $M_G = (EG \times M)/G$, where G acts freely on $EG \times M$. By Cartan, $H^*(M_G) = H^*\{(W(\mathfrak{g}) \otimes \Omega(EG \times M))_{\mathrm{bas}}\}$, this doesn't work because EG is not a manifold (it is infinite dimensional).

Then, in the more general case (not necesarily free action):

For a compact Lie group G, EG =infinite Stiefel variety $V(k, \infty)$, which can be apprximated by $V(k, n)$ for n sufficiently large, s.t. for any i, $H^i(V(k, n), \mathbb{R}) = H^i(V(k, \infty), \mathbb{R}) = 0$ for n sufficiently large.

$M_G = (EG \times M)/G$ can be approximated by $(EG(n) \times M)/G := M_G(n)$, so Cartan's theorem applies, and this gives the equivariant de Rham theorem in general.

Now let's go back to prove Cartan's theorem:

$0 \to \underbrace{\Omega(P)}_{=B} \hookrightarrow \underbrace{W(\mathfrak{g}) \otimes \Omega(P)_{\mathrm{bas}}}_{=\bar{B}} \to \bar{B}/B \to 0$ induces $\cdots \to H^*(\Omega(P)_{\mathrm{bas}}) \to H^*(\bar{B}) \to H^*(\bar{B}/B) \to \cdots$, where $H^*(\Omega(P)_{\mathrm{bas}}) = H^*(\Omega(N)) = H^*(N)$; $H^*(\bar{B}) = H^*(\text{Weil model})$. It turns out we can prove $H^*(\bar{B}/B) = 0$ (Cartan writes down a cochain homotopy between id and 0 to prove $H^*(\bar{B}/B) = 0$.)